ABSTRACT

Title of thesis: QUIVER REPRESENTATIONS AND AUSLANDER-REITEN THEORY

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A quiver is simply a directed graph studied algebraically. Despite their inherent graphical nature, quivers are not studied in the context of graph theory. Instead they are the tool of algebraic geometers and representation theorists.

In this thesis we present a self-contained expository note on the theory of quivers and their representations. Particularly; we examine the category $\operatorname{rep} Q$ of a fixed quiver Q and the variety of representations found within it, detail the categoric relationship between a quiver representation and a module over a finite-dimensional algebra, an extension to bound quivers and bound quiver algebras, and the establish Gabriel's theorem.

Chapter 3 introduces the representation theory of Artinian rings and almost split sequences through Auslander-Reiten theory. We further explore Auslander-Reiten theory by its relation with $\operatorname{rep} Q$ utilizing the Auslander-Reiten quiver Γ_Q, which acts as a map for the contemporary mathematician to traverse $\operatorname{rep} Q$. Focusing primarily on various methods of constructing Γ_Q.

Motivated by the geometric notions introduced whilst proving Gabriel's theorem Appendix A introduces the moduli spaces of quiver representations, which is at the heart of many areas of current algebraic geometry research. Furthermore, Appendix A.3 expands on this discussion to the very recent study of representation theory of neural networks.

Throughout this thesis we use the inherently visual nature of quivers to assist the reader in understanding the abstract notions of modules and categories which are discussed. Many examples are included, and for most examples included in Appendix B are computations done in GAP (through the QPA package) which follow the example, highlighting the use computational algebra systems have in modern mathematics.

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by

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Chapter 1: Introduction and Preliminaries

Quivers are an algebraist's interpretation of a directed graph. Formally, quivers were introduced by Gabriel in 1972 [Gab72], with the nomenclature arising as the diagrams are "boxes of arrows". Prior to Gabriel's formalization the study of such structures were described as diagram schemes by Grothendieck. They hold use in various fields and theories found across mathematics, and currently are at the forefront of algebraic geometry.

In this thesis we seek to explore quiver theory in a self-contained expository manner. Chapter 1 continues on to familiarize the reader with the concepts across algebra which are needed to understand the rest of the content. Much of the material presented throughout Chapter 1 is stated, and any theorems or results are often stated without proof. More in depth explanations can be found in, [Her96], [DF04], or [Hun12].

Chapter 2 introduces quivers formally, moving on then to discuss a variety of representations which occur over a fixed quiver Q, and the subsequent study of the category $\operatorname{rep} Q$. Here we present many examples illustrating the definitions, results, and concepts introduced. This chapter ultimately works towards a proof on the well-known and celebrated Gabriel's theorem, a result which spurred much of quiver theory. Most of this material was derived through a variety of sources including: [Sch14], [ASS06], [Uni12], [Bri08], and [Rei08].

Chapter 3 then shifts attention to Auslander-Reiten theory, and particularly the use of the Auslander-Reiten quiver Γ_Q which provides a map for the contemporary mathematician to traverse the category $\operatorname{rep} Q$ (and as we will later see, $\operatorname{mod} \Bbbk Q$). We highlight two unique algorithms for constructing Γ_Q, illustrating the construction of such a quiver in a variety of ways.

Appendix A provides further material on two related areas of quiver theory—moduli spaces of representations, and an application to the representation theory of neural networks. Lastly, in Appendix B one can find the appropriate GAP code which is used in parallel to the examples presented.

1.1 Preliminaries

Throughout this thesis we let $\Bbbk$ denote an (arbitrary) algebraically closed field, often in other literature $\Bbbk = \mathbb{C}$. We further assume familiarity with abstract algebra. Quivers themselves only require a familiarity with linear algebra, but some of the additional topics discussed will require a broader experience with module theory, ring theory, and geometry.

1.1.1 Category Theory

Category theory is a broad theory of mathematical structures and the relations between them. Presently, category theory can be found in most areas of mathematics, and in several areas of computer science.

Definition 1.1. A *category* $\mathcal{C} = (\operatorname{Ob}(\mathcal{C}), \operatorname{Hom}_{\mathcal{C}}, \circ)$ is a collection of objects and morphisms equipped with a binary operation called the composition of morphisms. We say $\operatorname{Ob}(\mathcal{C})$ is the class of objects of $\mathcal{C}$ and $\operatorname{Hom}_{\mathcal{C}}$ is the class of morphisms where a morphism $f \in \operatorname{Hom}_{\mathcal{C}}$ has a unique source and target object $f : X \to Y$. The class of all morphisms from X to Y is given as $\operatorname{Hom}_{\mathcal{C}}(X, Y)$.

For any three objects $X, Y, Z \in \mathrm{Ob}(\mathcal{C})$ we have that composition

$$\circ : \mathrm{Hom}_{\mathcal{C}}(X,Y) \times \mathrm{Hom}_{\mathcal{C}}(Y,Z) \to \mathrm{Hom}_{\mathcal{C}}(X,Z), \qquad \circ : (f,g) \mapsto g \circ f$$

satisfies the properties;

i. (Associativity) If $f : W \to X$, $g : X \to Y$, and $h : Y \to Z$ are morphisms then

$$h \circ (g \circ f) = (h \circ g) \circ f.$$

ii. (Identity) For all objects $X \in \mathrm{Ob}(\mathcal{C})$, there exists a morphism $1_X \in \mathrm{Hom}_{\mathcal{C}}(X,X)$ such that for all morphisms $f \in \mathrm{Hom}_{\mathcal{C}}(X,-)$ and $g \in \mathrm{Hom}_{\mathcal{C}}(-,X)$ we have

$$f \circ 1_X = f \qquad 1_X \circ g = g.$$

Note that in our definition we discuss categories in terms of classes and not sets, as this allows us to discuss the category of objects which cannot be classified as sets e.g., the category of all groups **GRP** which has the class of all groups as objects and the class of all group homomorphisms as morphisms.

Part of the beauty of category theory is the universal traits and properties found within categories. The notion of kernels is well-understood in the context of, say linear transformations, but these ideas generalize. The following is the universal property of kernels and cokernels.

Definition 1.2. Let $\mathcal{C}$ be a category with X, Y, Z as objects, furthermore let $g : Y \to Z$ be a morphism. Then a *kernel* of g is a morphism $f : X \to Y$ such that $gf = 0$ and for any given morphism $v : A \to Y$ such that $gv = 0$, there is a unique morphism

$u : A \to X$ such that $fu = v$. We say that v *factors through* f:

$$\begin{tikzcd} & A \arrow[dl, dashed, "\exists u"'] \arrow[d, "v"] & \\ X \arrow[r, "f"] & Y \arrow[r, "g"] & Z \end{tikzcd}$$

If, $f : X \to Y$ is a morphism then a *cokernel* of f is a morphism $g : Y \to Z$ such that $gf = 0$ and given any morphism $v : Y \to A$ such that $vf = 0$, there is a unique morphism $u : Z \to A$ such that $ug = v$. We say that v *factors through* g:

$$\begin{tikzcd} X \arrow[r, "f"] & Y \arrow[r, "g"] \arrow[dr, "v"'] & Z \arrow[d, dashed, "\exists u"] \\ & & A \end{tikzcd}$$

In the context of groups the First Isomorphism Theorem asserts that for given groups G and H with a group homomorphism $f : G \to H$ there is an isomorphism $G/\ker f \cong \operatorname{im} f$. This extends itself naturally to the diagram

$$\begin{tikzcd} & & & 0 \arrow[d] & \\ 0 \arrow[r] & \ker f \arrow[r, "\kappa"] & G \arrow[r, "\pi"] \arrow[dr, "f"'] & G/\ker f \cong \operatorname{im} f \arrow[r] \arrow[d, "\iota"] & 0 \\ & & & H & \end{tikzcd}$$

where π and ι are the natural projection and inclusion maps (of epimorphism and monomorphism type) and κ is a monomorphism. This diagram itself results in a *short exact sequence*, which will be discussed later.

We continue on with the discussion of categories by developing some further vocabulary. In linear algebra the set of all linear transformations between two $\Bbbk$-vector spaces, denoted as $\mathrm{Hom}(V, W)$, itself has a natural $\Bbbk$-vector space structure with addition and scaling of morphisms. If our chosen category $\mathcal{C}$ satisfies this property, i.e., $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ is a $\Bbbk$-vector space for all objects $X, Y \in \mathcal{C}$ then we say $\mathcal{C}$ is a $\Bbbk$-*category*.

Furthermore, a category is *additive* if it has direct sums and the existence of a zero object $0 \in \mathcal{C}$ such that the identity morphism $1_0 \in \mathrm{Hom}(0, 0)$ is the zero of the vector space $\mathrm{Hom}(0, 0)$.

If for any morphism $f : X \to Y \in \mathcal{C}$, there are kernels and cokernels i and p

$$W \xrightarrow{i = \ker f} X \xrightarrow{f} Y \xrightarrow{p = \operatorname{coker} f} Z$$

such that $\operatorname{coker} i \cong \ker p$, and $\mathcal{C}$ is an additive $\Bbbk$-category we say that $\mathcal{C}$ is an *abelian* $\Bbbk$-*category*.

We now formally introduce exact sequences, an integral notion fundamental for the results discussed throughout this thesis.

Definition 1.3. A sequence of morphisms $X \xrightarrow{f} Y \xrightarrow{g} Z$ is called *exact at Y* if $\operatorname{im} f = \ker g$. An sequence of morphisms

$$\cdots \longrightarrow X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} X_3 \xrightarrow{f_3} \cdots$$

is called *exact* if it is exact at each X_i. A *short exact sequence* is an exact sequence of the form

$$0 \longrightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow 0\ .$$

Note that a sequence is short exact if and only if f is injective, g is surjective, and $\operatorname{im} f = \ker g$. Naturally, for any morphism $f : X \to Y \in \mathcal{C}$ the sequence

$$0 \longrightarrow \ker f \xrightarrow{\iota} X \xrightarrow{f} Y \xrightarrow{\pi} \operatorname{coker} f \longrightarrow 0,$$

where ι and π are the natural inclusion and projection maps, is exact; and the sequence

$$0 \longrightarrow \ker f \xrightarrow{\iota} X \xrightarrow{g} \operatorname{coker} f \longrightarrow 0$$

is short exact.

Definition 1.4. A morphism $f : X \to Y$ is called a *section* if there exists a morphism $h : Y \to X$ such that $h \circ f = 1_X$.

A morphism $g : Y \to Z$ is called a *retraction* if there exists a morphism $h : Z \to Y$ such that $g \circ h = 1_Z$.

A short exact sequence

$$0 \longrightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow 0$$

is said to *split* if f is a section, or equivalently, if g is a retraction.

Functors

In the same way that a group homomorphism takes us from one group to another, or how a morphism takes us from one object to another in the same category, *functors* are the abstraction of morphisms applied to categories themselves.

Definition 1.5. Let $\mathcal{C}$ and $\mathcal{C}'$ be two $\mathbb{k}$-categories. A *covariant functor* $F : \mathcal{C} \to \mathcal{C}'$

is a mapping such that

$$F : (X \in \mathcal{C}) \mapsto (F(X) \in \mathcal{C}')$$

$$F : (f : X \to Y) \mapsto (F(f) : F(X) \to F(Y))$$

and $F(1_X) = 1_{F(X)}$ and $F(g \circ f) = F(g) \circ F(f)$.

A *contravariant functor* $F : \mathcal{C} \to \mathcal{C}'$ is a mapping such that

$$F : (x \in \mathcal{C}) \mapsto (F(X) \in \mathcal{C}')$$

$$F : (f : X \to Y) \mapsto (F(f) : F(Y) \to F(X))$$

and $F(1_X) = 1_{F(X)}$ and $F(g \circ f) = F(f) \circ F(g)$.

Two highly relevant functors are the Hom functors; $\mathrm{Hom}(X, -)$ and $\mathrm{Hom}(-, X)$ for a fixed object $X \in \mathcal{C}$.

We define the covariant $\mathrm{Hom}(X, -)$ functor to be the functor from the category $\mathcal{C}$ to the category of $\Bbbk$-vector spaces. This sends an object $Y \in \mathcal{C}$ to the vector space $\mathrm{Hom}(X, Y)$ of all morphisms between X and Y, it also sends a morphism $f : Y \to Z \in \mathcal{C}$ to the map $f_* : \mathrm{Hom}(X, Y) \to \mathrm{Hom}(X, Z)$, i.e., $f_*(g) = f \circ g$:

$$\begin{tikzcd}
X \arrow[d, "g"'] \arrow[dr, "f_*"] & \\
Y \arrow[r, "f"] & Z
\end{tikzcd}$$

We similarly define the contravariant $\mathrm{Hom}(-, X)$ functor to be the functor from the category $\mathcal{C}$ to the category of $\Bbbk$-vector spaces. However, this sends an object $Y \in \mathcal{C}$ to the vector space $\mathrm{Hom}(Y, X)$ of all morphisms between Y and X, it also sends a morphism $f : Y \to Z \in \mathcal{C}$ to the map $f^* : \mathrm{Hom}(Y, X) \to \mathrm{Hom}(Z, X)$, i.e.,

$f^*(g) = g \circ f$:

$$\begin{tikzcd} Y \arrow[r, "f"] \arrow[dr, "f^*"'] & Z \arrow[d, "g"] \\ & X \end{tikzcd}$$

The maps f_* and f^* are called the *push forward* and *pull back* of f respectively.

Definition 1.6. If $\mathcal{C}$ and $\mathcal{D}$ are two categories we say that two functors $F_1, F_2 : \mathcal{C} \to \mathcal{D}$ are *functorially isomorphic*, denoted as $F_1 \cong F_2$, if for every object $X \in \mathcal{C}$ there exists an isomorphism $\eta_X : F_1(X) \to F_2(X) \in \mathcal{D}$ such that for every morphism $f : X \to Y \in \mathcal{C}$ the following diagram commutes:

$$\begin{tikzcd} F_1(X) \arrow[r, "F_1(f)"] \arrow[d, "\eta_X"'] & F_1(Y) \arrow[d, "\eta_Y"] \\ F_2(X) \arrow[r, "F_2(f)"] & F_2(Y) \end{tikzcd}$$

Definition 1.7. A covariant functor $F : \mathcal{C} \to \mathcal{D}$ is called an *equivalence of categories* if there exists a functor $G : \mathcal{D} \to \mathcal{C}$ such that $G \circ F \cong 1_{\mathcal{C}}$ and $F \circ G \cong 1_{\mathcal{D}}$. Such a functor G is called a *quasi-inverse functor* for F.

For two abelian categories $\mathcal{C}$ and $\mathcal{D}$, a functor between the two is called *exact* if it maps exact sequences in $\mathcal{C}$ to exact sequences in $\mathcal{D}$.

And lastly, if a contravariant functor F has a contravariant quasi-inverse then we say F is a *duality*.

Simple, Projective, and Injective Objects

Projective and injective objects are key concepts in general category theory. They are also of particular interest in representation theories, which will further be investigated in Section 2.1.3.

An object $\mathcal{P} \in \mathcal{C}$ is called *projective* if the covariant functor, $\mathrm{Hom}(\mathcal{P}, -)$ maps surjective morphisms to surjective morphisms. More explicitly, for any object X there is some projective object $\mathcal{P}_0$ such that there is a surjective morphism;

$$p_0 : \mathcal{P}_0 \twoheadrightarrow X,$$

which "projects" P_0 to X.

The dual notion of projectivity is injectivity, an object $\mathcal{I}$ is called *injective* if the contravariant functor, $\mathrm{Hom}(-, \mathcal{I})$, maps injective morphisms to injective morphisms. For any object X there is some injective representation $\mathcal{I}_0$ such that there exists an injective morphism;

$$i_0 : X \hookrightarrow \mathcal{I}_0,$$

which "injects" X to $\mathcal{I}_0$.

Definition 1.8. For a non-projective object X the morphism p_0 above has a kernel and we can find another projective $\mathcal{P}_1$ such that the surjective map $p_1 : \mathcal{P}_1 \to \ker p_0 = \mathcal{P}_0$. Repeating this results in the exact sequence

$$\cdots \longrightarrow \mathcal{P}_2 \xrightarrow{p_2} \mathcal{P}_1 \xrightarrow{p_1} \mathcal{P}_0 \xrightarrow{p_0} X \longrightarrow 0 \ ,$$

called a *projective resolution of* X. Dually, an exact sequence of the form

$$0 \longrightarrow X \xrightarrow{i_0} \mathcal{I}_0 \xrightarrow{i_1} \mathcal{I}_1 \xrightarrow{i_2} \mathcal{I}_2 \longrightarrow \cdots \ ,$$

is called an *injective resolution*.

A *simple* object is a non-zero object $\mathcal{S}$ with no proper subobjects. We state the following result but omit a proof of it as it is a standard result.

Theorem 1.1. *Let $\mathcal{C}$ be any additive category;*

1. *If $\mathcal{P}, \mathcal{P}' \in \mathcal{C}$ are two objects then $\mathcal{P} \oplus \mathcal{P}'$ is projective if and only if $\mathcal{P}$ and $\mathcal{P}'$ are both projective.*

2. *If $\mathcal{I}, \mathcal{I}' \in \mathcal{C}$ are two objects then $\mathcal{I} \oplus \mathcal{I}'$ is projective if and only if $\mathcal{I}$ and $\mathcal{I}'$ are both injective.*

Continuing on with the ideas of projective and injective resolutions we wish to introduce the idea of minimality of such resolutions. To do so however we first need covers and envelopes.

Definition 1.9. Let X be an object in $\mathcal{C}$. A *projective cover* of X is a projective object $\mathcal{P}$ together with a surjective morphism $g : \mathcal{P} \to X$ with the added property that whenever $g' : \mathcal{P}' \to X$ is a surjective morphism with $\mathcal{P}'$ projective, there exists a *surjective* morphism $h : \mathcal{P}' \twoheadrightarrow \mathcal{P}$ such that the diagram

$$\begin{tikzcd}
& \mathcal{P}' \arrow[dl, dashed, two heads, "h"'] \arrow[d, "g'"] & \\
\mathcal{P} \arrow[r, "g"] & X \arrow[r] \arrow[d] & 0 \\
& 0 &
\end{tikzcd}$$

commutes, i.e. $gh = g'$.

An *injective envelope* of X is an injective object $\mathcal{I}$ with an injective morphism $f : X \to \mathcal{I}$ with the added property that whenever $f' : X \to \mathcal{I}'$ is an injective morphism with $\mathcal{I}'$ injective, there exists an *injective* morphism $h : \mathcal{I} \to \mathcal{I}'$ such that the diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & \\
0 \arrow[r] & X \arrow[r, "f"] \arrow[d, "f'"'] & \mathcal{I} \arrow[dl, dashed, hook, "h"] \\
 & \mathcal{I}' &
\end{tikzcd}$$

commutes, i.e. $hf = f'$.

Definition 1.10. A projective resolution of X

$$\cdots \longrightarrow \mathcal{P}_2 \xrightarrow{p_2} \mathcal{P}_1 \xrightarrow{p_1} \mathcal{P}_0 \xrightarrow{p_0} X \longrightarrow 0\ ,$$

is called *minimal* if $p_0 : \mathcal{P}_0 \to X$ is a projective cover and $p_n : \mathcal{P}_n \to \ker p_{n-1}$ is a projective cover for all $n > 0$. Similarly an injective resolution of X

$$0 \longrightarrow X \xrightarrow{i_0} \mathcal{I}_0 \xrightarrow{i_1} \mathcal{I}_1 \xrightarrow{i_2} \mathcal{I}_2 \longrightarrow \cdots\ ,$$

is called *minimal* if $i_0 : X \to \mathcal{I}_0$ is an injective envelope and $i_n : \operatorname{coker} i_{n-1} \to \mathcal{I}_n$ is an injective envelope for all $n > 0$.

We now introduce the last category theoretic notion needed which is the Ext-groups. Let X be any object, and take a projective resolution of X in $\mathcal{C}$, say of the form,

$$0 \longrightarrow \mathcal{P}_1 \xrightarrow{f} \mathcal{P}_0 \xrightarrow{g} X \longrightarrow 0\ ,$$

For any other object $Y \in \mathcal{C}$ apply the functor $\operatorname{Hom}(-, Y)$ to the above resolution.

This results in the exact sequence

$$0 \longrightarrow \mathrm{Hom}(X,Y) \xrightarrow{g^*} \mathrm{Hom}(\mathcal{P}_0,Y) \xrightarrow{f^*} \mathrm{Hom}(\mathcal{P}_1,Y) \longrightarrow \mathrm{Ext}^1(X,Y) \longrightarrow 0$$

where $\mathrm{Ext}^1(X,Y) = \mathrm{coker}\, f^*$ is the *first group of extensions* of X and Y.

In general, projective resolutions however are not of the form above, instead generally they take the form

$$0 \longrightarrow \mathcal{P}_n \xrightarrow{f_n} \mathcal{P}_{n-1} \xrightarrow{f_{n-1}} \cdots \xrightarrow{f_2} \mathcal{P}_1 \xrightarrow{f_1} \mathcal{P}_0 \xrightarrow{f_0} X \longrightarrow 0$$

which when the $\mathrm{Hom}(-,Y)$ functor is applied yields a *co-chain complex*

$$0 \longrightarrow \mathrm{Hom}(X,Y) \xrightarrow{f_0^*} \mathrm{Hom}(\mathcal{P}_0,Y) \xrightarrow{f_1^*} \cdots \xrightarrow{f_n^*} \mathrm{Hom}(\mathcal{P}_n,Y) \longrightarrow \cdots ,$$

where $f_i^* f_{i-1}^* = 0$ for all i. The, the ith extension group $\mathrm{Ext}^i(X,Y)$ for $i \geq 1$ is the ith cohomology group of this complex. That is,

$$\mathrm{Ext}^i(X,Y) = \ker f_{i+1}^* / \mathrm{im}\, f_i^*.$$

Definition 1.11. An extension ζ of X by Y is a short exact sequence

$$0 \longrightarrow X \xrightarrow{f} E \xrightarrow{g} Y \longrightarrow 0 .$$

Two extensions ζ and ζ' are *equivalent* if there is a commutative diagram:

$$\begin{tikzcd}
\zeta\colon & 0 \arrow[r] & X \arrow[r, "f"] \arrow[d, equal] & E \arrow[r, "g"] \arrow[d, rightsquigarrow] & Y \arrow[r] \arrow[d, equal] & 0 \\
\zeta'\colon & 0 \arrow[r] & X \arrow[r, "f'"] & E' \arrow[r, "g'"] & Y \arrow[r] & 0
\end{tikzcd}$$

1.1.2 Algebra

In this section we recall some definitions from linear algebra, group theory, ring theory, and representation theory developing a familiarity with algebras and modules. Familiarity with elementary abstract algebra is assumed. Throughout this section assume G is a group and R is a ring with unit. We use by default e to mean the identity in groups and 1 the multiplicative identity for rings.

Definition 1.12. Let G be a group and X be any set, then a *group action* of G on X is a binary function $G \times X \to X$, denoted as $(g, x) \mapsto gx$, such that for all $g, g' \in G$ and $x \in X$;

1. $(gh)x = g(hx)$,
2. and $ex = x$.

Such a set X is then called a G-set. For an element $x \in X$ we define the *orbit of* x to be $\mathcal{O}_x = G \cdot x := \{g \cdot x \mid g \in G\}$. The *stabilizer* of x is given as $\operatorname{Stab} x := \{g \in G \mid g \cdot x = x\}$ and forms a subgroup of G.

We recall some basic representation theory of finite groups. Further reading can be found in [DF04] or [Fei82].

Definition 1.13. Let G be a group and V be a $\mathbb{k}$-vector space, then a *representation* of G is a group homomorphism $\rho : G \to \mathrm{GL}(V)$ where $\mathrm{GL}(V)$ is the general linear group on V.

The representation space of ρ is V and the dimension of the representation is given as $\dim V$. We define a *character* of a group representation ρ to be the map $\chi_\rho : G \to \mathbb{k}$ where $\chi_\rho : g \mapsto \mathrm{Tr}(\rho(g))$. The underlying idea here is to translate an algebraic object, in this case a (finite) group, to the language of linear algebra since linear algebra is a traditionally well-understood field.

Recall the notion of ideals and general ring theory.

Definition 1.14. Let R be a ring, a nonempty subset $I \subset R$ is an *ideal* of R if I is an additive subgroup of R, i.e., $(I, +) \leq (R, +)$ and for any $r \in R$ and $a \in I$ we have $ra \in I$ (left ideal), $ar \in I$ (right ideal), or $ra, ar \in I$ (two-sided ideal).

If $I \subset R$ is a proper ideal then we say I is *maximal* if the only ideals that contain I are I and R itself. We say I is *nilpotent* if there exists some $m \geq 1$ such that $I^m = 0$.

Definition 1.15. The *Jacobson radical* $\mathrm{rad}\, R$ is the intersection of all maximal right ideals in R.

A ring that contains only one maximal right ideal is called a *local ring.* In local rings the unique maximal right ideal is equal to the radical.

Definition 1.16. A $\mathbb{k}$-*algebra* A is a ring $(A, +, \cdot)$ with unity such that A has a $\mathbb{k}$-vector space structure such that;

i. Addition in the vector space A is the same as in the ring A.

ii. Scalar multiplication in the vector space A is compatible with the ring multiplication, that is for all $a, b \in A$ and $\lambda \in \Bbbk$,

$$\lambda(ab) = (\lambda a)b = a(\lambda b) = (ab)\lambda.$$

The *dimension* of A (the algebra) is the dimension of A (the vector space). As with rings A is said to be *local* if it has a unique maximal right ideal.

An element $e \in A$ is *idempotent* if $e^2 = e$. An idempotent is *central idempotent* if $ea = ae$ for all $a \in A$. Two idempotents e, e' are *orthogonal* if $ee' = e'e = 0$. A non-zero idempotent e is called *primitive* if e cannot be written as $e = e_1 + e_2$ where e_1, e_2 are non-zero orthogonal idempotents. For any algebra A, 0 and 1 are trivial idempotents.

We now shift to some basic module theory.

Definition 1.17. Let R be a ring with unity, a right R-module M is an abelian group equipped with a binary operation called right R-action, $M \times R \to M$ such that $(m, r) \mapsto mr$. With the following properties holding for all $m, n \in M$ and $r, s \in R$;

i. $(m + n)r = mr + nr$,

ii. $m(r + s) = mr + ms$,

iii. $m(rs) = (mr)s$, and

iv. $m1 = m$.

One can similarly define a left module (with a left R-action), and it should be noted that a $\Bbbk$-vector space V is in fact a left-right $\Bbbk$-module. Common examples of modules would be R itself or $I \leq R$ a right ideal of R. If M is an R-module and I is a right ideal then the set $MI := \{m_1r_1+\cdots+m_tr_t \mid m_i \in M, r_i \in I\}$ is a sub-module of M.

A module is said to be *generated* by the elements $m_1, \dots, m_t$ if for every $m \in M$ there exists $a_i \in R$ such that $m = m_1a_1+\cdots+m_ta_t$. If the set of generating elements is finite, then we say M is finitely-generated.

For two R-modules, M and N a map $h : M \to N$ is a *morphism* of R-modules if for all $m, m' \in M$ and $r \in R$ we have

$$h(m+m') = h(m)+h(m'), \qquad h(ma) = h(m)a.$$

For any R-module morphism h we may define the new R-modules; $\ker h := \{m \in M \mid h(m) = 0\}$, $\operatorname{im} h := \{h(m) \mid m \in M\}$, and $\operatorname{coker} h := N/\operatorname{im} h$. Where $\ker h$ and $\operatorname{im} h$ are sub-modules of M and N respectively, as one would expect.

If M is an $A-$*module* for some $\Bbbk$-algebra A, then the set of all morphisms $M \to M$, called *endomorphisms* $\operatorname{End} M$, has a $\Bbbk$-vector space structure just as in $\operatorname{Hom}(-,-)$. This allows us to define the $\Bbbk$-algebra of all endomorphisms of M, $\operatorname{End} M$.

We now list two fundamental results in module theory, but omit their proofs.

Lemma 1.2 (Nakayama's Lemma)**.** *Let M be a finitely-generated R-module and $I \subseteq \operatorname{rad} R$ a two-sided ideal contained in the radical of R. If $MI = M$ then $M = 0$.*

Lemma 1.3 (Five Lemma)**.** *Given a commutative diagram of R-modules with exact rows:*

$$\begin{tikzcd}
M_1 \arrow[r, "f_1"] \arrow[d, "\varphi_1"] & M_2 \arrow[r, "f_2"] \arrow[d, "\varphi_2"] & M_3 \arrow[r, "f_3"] \arrow[d, "\varphi_3"] & M_4 \arrow[r, "f_4"] \arrow[d, "\varphi_4"] & M_5 \arrow[d, "\varphi_5"] \\
N_1 \arrow[r, "g_1"] & N_2 \arrow[r, "g_2"] & N_3 \arrow[r, "g_3"] & N_4 \arrow[r, "g_4"] & N_5
\end{tikzcd}$$

Then

1. *If φ_2 and φ_4 are surjective and φ_5 is injective, then φ_3 is surjective.*

2. *If φ_1 is surjective and φ_2 and φ_4 are injective, then φ_3 is injective.*

3. *If φ_1 is surjective, φ_2 and φ_4 are isomorphisms, and φ_5 is injective, then φ_3 is an isomorphism.*

The following definitions refer to the *homological dimension* for modules over algebras. Intuitively, the projective (resp. injective) dimension of a module is a measure of "how far a module is from being projective" (resp. injective). The global dimension is a measure of "how far is the algebra from being hereditary".

Definition 1.18. Let M be an A-module, the *projective dimension* $\mathrm{pd}(M)$ is the smallest integer d such that

$$0 \longrightarrow \mathcal{P}_d \longrightarrow \cdots \longrightarrow \mathcal{P}_1 \longrightarrow \mathcal{P}_0 \longrightarrow M \longrightarrow 0$$

is a projective resolution. If no such resolution exists we say that M has infinite projective dimension. We then define the *global dimension* of M to be

$$\operatorname{gldim} M := \sup\{\mathrm{pd}(M) \mid M \in \operatorname{mod} A\}.$$

The *injective dimension* $\mathrm{id}(M)$ is the smallest integer d such that

$$0 \longrightarrow M \longrightarrow \mathcal{I}_0 \longrightarrow \mathcal{I}_1 \longrightarrow \cdots \longrightarrow \mathcal{I}_d \longrightarrow 0$$

is an injective resolution. If no such resolution exists we say that M has infinite injective dimension.

Chapter 2: Quivers and Quiver Representations

A quiver is a directed graph and a representation of that quiver is an association of vector spaces to vertices and linear maps to arrows. Quivers and their representations are relatively simple objects but they play a vital role in the representation theory of finite-dimensional algebras.

Section 2.1 provides a more algebraic approach to quivers and their representations, followed by Section 2.2 which introduces the associated path algebra. In Section 2.2.1 the categories of finite-dimensional quiver representations and finite-dimensional $\Bbbk Q$-modules are introduced, and their equivalence is proven. A more geometric interpretation is given en route to proving Gabriel's theorem in Section 2.3. A few familiar results are also shown.

Formally, we define a quiver as follows.

Definition 2.1. A *quiver* $Q = (Q_0, Q_1, s, t)$ is a quadruple consisting of; Q_0 a set of *vertices*, Q_1 a set of *arrows*, and two mappings $s, t : Q_1 \to Q_0$ which take and arrow to its *source* and *target* vertex respectively.

Quivers are quite expressive, they can contain infinitely many vertices or arrows, cycles, self-loops, etc… However, our interest lies primarily with *finite quivers* (i.e., $|Q_0| \neq \infty$ and $|Q_1| \neq \infty$). For the duration of this thesis a quiver is assumed to be finite unless otherwise stated. Other standard ideas from graph theory translate naturally, for instance a quiver is said to be connected if the *underlying diagram* (or *underlying graph*, denoted as Δ_Q and achieved by replacing arrows with edges) itself is connected.

We assume standard quiver notation, where vertices are enumerated with positive integers and arrows are labelled with greek letters.

Example 2.2. Let $Q = (Q_0, Q_1, s, t)$ be defined with $Q_0 = \{1, 2, 3\}$, $Q_1 = \{\alpha, \beta\}$, and

$$s : \alpha \mapsto 1, \quad t : \alpha \mapsto 2 \qquad s : \beta \mapsto 3, \quad t : \beta \mapsto 2.$$

Then, one visualization of Q could be

$$1 \xrightarrow{\alpha} 2 \xleftarrow{\beta} 3\ . \tag{2.1}$$

A familiar quiver is the *Kroenecker* quiver of the form

$$\begin{tikzcd} 1 \arrow[r, shift left, "\alpha"] \arrow[r, shift right, "\beta"'] & 2 \end{tikzcd}\ .$$

This particular quiver generalizes to the Kroenecker r-quiver, which contains r arrows between two vertices.

2.1 Representations

Representation theory seeks to translate an algebraic structure to the language of linear algebra. In finite group representation theory, one associates to a group G the map $\rho : G \to \mathrm{GL}(V)$ where V is a $\Bbbk$-vector space and ρ is called a representation of G. In quiver theory, one creates a representation naturally.

Definition 2.3. Let $Q = (Q_0, Q_1, s, t)$ be a quiver, a *representation*

$$M = (M_i, \varphi_\alpha)_{i \in Q_0, \alpha \in Q_1}$$

of Q is an association of vector spaces to vertices and linear maps to arrows. That is, M is a collection of $\Bbbk$-vector spaces and linear maps between them.

A representation $M = (M_i, \varphi_\alpha)$ is said to be *finite-dimensional* if each vector space M_i is finite dimensional. We also define the *dimension vector* $\underline{\dim} M = (\dim M_i)$ to be the n-tuple of dimensions for each vector space in the collection.

As with other algebraic structures, we have the natural notion of morphisms.

Definition 2.4. Let Q be a quiver with representations $M = (M_i, \varphi_\alpha)$ and $N = (N_i, \psi_\alpha)$. A *morphism* between representations $f : M \to N$ is a collection of linear maps $f_i : M_i \to N_i$ such that for all arrows $i \xrightarrow{\alpha} j \in Q_1$ the following diagram commutes

$$\begin{tikzcd}
M_i \arrow[r, "\varphi_\alpha"] \arrow[d, "f_i"'] & M_j \arrow[d, "f_j"] \\
N_i \arrow[r, "\psi_\alpha"] & N_j
\end{tikzcd}$$

2.1.1 Direct Sums and Indecomposable Representations

Continuing with our definitions we now introduce direct sums and the notion of indecomposable representations.

Definition 2.5. Let Q be a quiver with representations $M = (M_i, \varphi_\alpha)$ and $N =$

(N_i, ψ_α). Then,

$$M \oplus N = \left(M_i \oplus N_i, \left[\begin{array}{c|c} \varphi_\alpha & 0 \\ \hline 0 & \psi_\alpha \end{array} \right] \right)_{i \in Q_0, \alpha \in Q_1}$$

is another representation of Q, called the *direct sum* of M and N.

A representation M of Q is said to be *indecomposable* if there are no non-zero representation N and L such that $M = N \oplus L$. Indecomposable representations act as the building blocks of other representations, in a sense they are analogous to primes to the integers.

The following example illustrates the three previous concepts.

Example 2.6. Suppose we have Q as in 2.1. We first define two representations of Q as follows;

$$L: \qquad \Bbbk^2 \xrightarrow{\quad T \quad} \Bbbk^2 \xleftarrow{\quad S \quad} \Bbbk$$

$$M: \qquad \Bbbk \xrightarrow{\quad 1 \quad} \Bbbk \xleftarrow{\quad 0 \quad} 0$$

with maps T and S given as

$$T = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = 2I \qquad S = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

We can now consider a morphism of representations $f : L \to M$, illustrated below as dashed arrows.

$$\begin{tikzcd}
L \arrow[d, dashed, "f"'] & \Bbbk^2 \arrow[r, "T"] \arrow[d, dashed, "f_1"] & \Bbbk^2 \arrow[d, dashed, "f_2"] & \Bbbk \arrow[l, "S"'] \arrow[d, dashed, "f_3"] \\
M & \Bbbk \arrow[r, "1"] & \Bbbk & 0 \arrow[l, "0"']
\end{tikzcd}$$

It follows then, that $f_1 : \Bbbk^2 \to \Bbbk$ is of the form $\begin{bmatrix} a & b \end{bmatrix}$, likewise $f_2 : \Bbbk^2 \to \Bbbk$ has form $\begin{bmatrix} c & d \end{bmatrix}$, and lastly $f_3 : \Bbbk \to 0$ is just the zero map. We then get the relations $a + b = c + d = 0$ and thus $b = -a$ and $d = -c$. However we also have the added condition that $a = 2c$, this follows from the need to have the diagram commute, and $T = 2I$. Since one choice of scalar $a \in \Bbbk$ completely determines the morphism we see that $\mathrm{Hom}(L, M) \cong \Bbbk$.

Now consider a new representation $N \cong L \oplus M$, first we compute $L \oplus M$

$$L \oplus M: \qquad \Bbbk^2 \oplus \Bbbk \xrightarrow{\left[\begin{array}{cc|c} 2 & 0 & 0 \\ 0 & 2 & 0 \\ \hline 0 & 0 & 1 \end{array}\right]} \Bbbk^2 \oplus \Bbbk \xleftarrow{\left[\begin{array}{c|c} 1 & 0 \\ 1 & 0 \\ \hline 0 & 0 \end{array}\right]} \Bbbk \oplus 0 \ .$$

Thus, we see that N is the representation

$$N: \qquad \Bbbk^3 \xrightarrow{\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \Bbbk^3 \xleftarrow{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}} \Bbbk \ .$$

Broadly, across various representation theories the goal is to classify all representations of a structure and the morphisms between them up to isomorphism. We now present the well-known Krull-Schmidt theorem (in the context of quivers). This result tells us that it suffices to study only indecomposable representations to classify all representations.

Theorem 2.1 (Krull-Schmidt). *Let Q be a quiver and M a representation of Q, then*

$$M \cong M_1 \oplus M_2 \oplus \cdots \oplus M_t$$

where the M_i are indecomposable representations of Q that are unique up to order.

Proof. We prove existence only as the proof of uniqueness requires a series of previous results, we reference [Hun12, Chapter 10] for a full proof.

If M is indecomposable there is nothing to show. Suppose then that M is not indecomposable, then $M \cong N \oplus L$ where $\underline{\dim} N$ and $\underline{\dim} L$ are strictly less than $\underline{\dim} M$. By induction we see that $N \cong N_1 \oplus N_2 \oplus \cdots N_t$ and $L \cong L_1 \oplus L_2 \oplus \cdots L_s$ where the N_i and L_j are all indecomposable. This proves existence of the desired decomposition. $\square$

2.1.2 Kernels and Cokernels of Morphisms

Suppose V and W are two $\Bbbk$-vector spaces with $T : V \to W$ a linear map between them. It is known that $\ker T := \{v \in V \mid T(v) = 0\}$ is a subspace of V and $\operatorname{coker} T := \{w + f(v) \mid v \in V\}$ is a quotient space of W. In this subsection we generalize these ideas to representations of quivers. We will further introduce exact sequences of representations in the context of quivers.

Fix Q to be a quiver, with representations $M = (M_i, \varphi_\alpha)$ and $M' = (M'_i, \varphi'_\alpha)$ and morphism $f : M \to M'$.

Definition 2.7. Let $L_i := \ker f_i$ be a vector space for each vertex, note that $L_i \leq M_i$. For each arrow $i \xrightarrow{\alpha} j \in Q_1$ let $\psi_\alpha : L_i \to L_j$ be a linear map between the two spaces given as $\varphi_\alpha |_{L_i}$. That is, ψ_α is the restriction of φ_α to L_i. Then, $\ker f = (L_i, \psi_\alpha)$ is a representation of Q called the *kernel of f*.

To show that $\ker f$ is in fact a representation it suffices to verify that ψ_α is well-defined. Let $l \in L_i$ we need to show that $\psi_\alpha(l) \in L_j$, i.e., $\varphi_\alpha(l) \in \ker f_j$. Since f is a morphism of representations we have that $f_j \varphi_\alpha(l) = \varphi'_\alpha(l) f_i = 0$ since $l \in \ker f_i$. Thus ψ_α is well-defined.

Definition 2.8. Let $N_i := \operatorname{coker} f_i = M'_i / f_i(M_i)$ be a vector space for each vertex. For each arrow $i \xrightarrow{\alpha} j \in Q_1$ let $\chi_\alpha : N_i \to N_j$ be defined by

$$\chi_\alpha : m'_i + f_i(M_i) \mapsto \varphi'_\alpha(m'_i) + f_j(M_j).$$

for all $m'_i \in M'_i$. Then, $\operatorname{coker} f = (N_i, \chi_\alpha)$ is a representation of Q called the *cokernel of f*.

We now show that χ_α is well-defined. Suppose there are two elements $m_1, m_2 \in M'_i$ such that $m_1 + f(M_i) = m_2 + f(M_i)$ then $m_1 - m_2 \in f(M_i)$. Thus,

$$\varphi'_\alpha(m_1) - \varphi'_\alpha(m_2) = \varphi'_\alpha(m_1 - m_2) \in \varphi'_\alpha f_i(M_i) = f_j \varphi_\alpha(M_i) \subset f_j(M_j).$$

So $\chi_\alpha(m_1 + f_i(M_i)) = \chi_\alpha(m_2 + f_i(M_i))$ and χ_α is well-defined.

The natural inclusions $\operatorname{incl}_i : \ker f_i \hookrightarrow M_i$ induce an injective morphism of representations

$$\operatorname{incl} : \ker f \hookrightarrow M.$$

Similarly, the natural projections $\operatorname{proj}_i : M'_i \twoheadrightarrow \operatorname{coker} f_i$ induce a surjective morphism of representations

$$\mathrm{proj} : M' \twoheadrightarrow \operatorname{coker} f.$$

One can see that these definitions of kernel and cokernel both satisfy the universal properties mentioned in Section 1.1.1.

Definition 2.9. If L and M are representations of a quiver such that there is some injective morphism $i : L \hookrightarrow M$, then L is a *subrepresentation* of M. We can also define the *quotient representation* M/L to be $\operatorname{coker} i$.

One can state the first isomorphism theorem with regards to a morphism $f : M \to N$ of quiver representations by $\operatorname{im} f \cong M/\ker f$.

These definitions, along with the notions from Section 1.1.1, paint a clearer picture of the collection of all representations of Q—$\operatorname{rep} Q$. Particularly that $\operatorname{rep} Q$ is an abelian $\Bbbk$-category. For any representations $M, N \in \operatorname{rep} Q$ with morphism $f : M \to N$ we have the following exact sequence

$$0 \longrightarrow \ker f \longrightarrow M \xrightarrow{f} N \longrightarrow \operatorname{coker} f \longrightarrow 0$$

and from this one can form the short exact sequence

$$0 \longrightarrow \ker f \longrightarrow M \longrightarrow M/\ker f \longrightarrow 0 \ .$$

Example 2.10. Let Q be the quiver in Example 2.6, representations $L, M \in \operatorname{rep} Q$ as defined previously, and some morphism $g : L \to M$ (recall that any morphism $g : L \to M$ is determined by one scalar $a \in \Bbbk$, e.g. $g = (g_1, g_2, g_3)$ with $g_1 : (x, y) \mapsto (ax - ay)$, $g_2 : (x, y) \mapsto \frac{1}{2}(ax-ay)$, and $g_3 : x \mapsto 0$). Suppose that we choose $a = 10 \in \Bbbk$ (suppose that $\Bbbk$ has characteristic greater than 10). Now consider $\ker g \in \operatorname{rep} Q$, it follows that $\underline{\dim} \ker g = (1, 1, 1)$ and moreover, $f = \mathrm{incl} : \ker g \hookrightarrow L$ is the natural inclusion which results in an injective morphism. For $\operatorname{coker} g \in \operatorname{rep} Q$, it is less intuitive to see, but $\operatorname{coker} g = 0$. Thus we get the short exact sequence

$$0 \longrightarrow \ker g \xrightarrow{\quad f \quad} L \xrightarrow{\quad g \quad} M \longrightarrow \operatorname{coker} g = 0 \ .$$

Remark. For any category $\mathcal{C}$, there is a *forgetful functor*, a functor which "drops" some of the input's properties/structure, $F : \mathcal{C} \to \mathbf{Quiv}$ which sends the category to a quiver.

2.1.3 Simple, Projective, and Injective Representations

In Section 1.1.1 we introduced projective and injective representations, here we define them in the context of quiver representations. Throughout the rest of this thesis much of our derived results and material relies on projective, specifically indecomposable projective, representations.

We begin with the formalization of paths in quivers.

Definition 2.11. Let Q be a quiver with vertices i and j in Q_0. A *path* of length ℓ between i and j is a sequence $c = (i \mid \alpha_1, \alpha_2, \ldots, \alpha_\ell \mid j)$. Where $\alpha_n \in Q_1$ such that, $s(\alpha_1) = i$, $t(\alpha_\ell) = j$, and for $2 \leq n \leq \ell$, $s(\alpha_n) = t(\alpha_{n-1})$.

There are a variety of common paths which will be of use to us later. We define the *constant path* at vertex i by $(i \mid\mid i)$ with length $\ell = 0$, we use e_i to denote this particular path. An arrow $i \xrightarrow{\alpha} i$ has a path $(i \mid \alpha \mid i)$ of length $\ell = 1$ called a *loop*. If a path is of the form $(i \mid \alpha_1, \alpha_2, \ldots, \alpha_\ell \mid i)$ then it is an *orientated cycle* of length ℓ.

Fix a vertex i in Q.

Definition 2.12. A *simple representation* $\mathcal{S}(i) \in \operatorname{rep} Q$ is such that the dimension at each vertex is zero except at vertex i, where it has dimension one. And set the maps to be 0 for all arrows α. That is, $\mathcal{S}(i) = (\mathcal{S}(i)_j, \varphi_\alpha)$ where

$$\mathcal{S}(i)_j := \begin{cases} \Bbbk & i = j \\ 0 & i \neq j \end{cases}$$

Computing $\mathcal{S}$ for a quiver is trivial, rather than detailing any particular example we point to Figure 2.1 where you can see all representations for Q as in (2.2). We now discuss both projective and injective representations, these are particularly important. Conventionally we will often discuss just projective representations as many of the results and notions for projective representations have a dual for injective representations.

Definition 2.13. A *projective representation* $\mathcal{P}(i) \in \operatorname{rep} Q$ is such that the dimension at each vertex j is the number of paths from i to j. If $j \xrightarrow{\alpha} l$ is an arrow in Q, then $\varphi_\alpha : \mathcal{P}(i)_j \to \mathcal{P}(i)_l$ is defined by composing the arrow α to the paths from $i \to j$. That is the injective map f acts on basis elements as,

$$f : c = (i \mid \beta_1, \ldots, \beta_t \mid j) \mapsto c\alpha = (i \mid \beta_1, \ldots, \beta_t, \alpha \mid l).$$

Since the basis of each $\mathcal{P}(i)_j$ is the set of all paths from $i \to j$ we have that

$$\varphi_\alpha : \mathcal{P}(i)_j \to \mathcal{P}(i)_l \qquad \varphi_\alpha : \sum \lambda c \mapsto \sum \lambda c\alpha.$$

Then, $\mathcal{P}(i) = (\mathcal{P}(i)_j, \varphi_\alpha)$ is the projective representation at vertex i.

Definition 2.14. An *injective representation* $\mathcal{I}(i) \in \operatorname{rep} Q$ is such that the dimension at each vertex j is the number of paths from j to i. If $j \xrightarrow{\alpha} l$ is an arrow in Q, then $\varphi : \mathcal{I}(i)_j \to \mathcal{I}(i)_l$ is defined by removing the arrow α from the paths $j \to i$ which start with α (if α is not the first arrow then we set that path to 0). That is the surjective map g acts on basis elements as,

$$g : c = (j \mid \beta_1, \ldots, \beta_t \mid i) \mapsto \begin{cases} c = (l \mid \beta_2, \ldots, \beta_t \mid i) & \beta_1 = \alpha \\ 0 & \beta_1 \neq \alpha \end{cases}.$$

Since the basis of each $\mathcal{I}(i)_j$ is the set of all paths from $i \to j$ we have that

$$\varphi_\alpha : \mathcal{I}(i)_j \to \mathcal{I}(i)_l \qquad \varphi : \sum \lambda c \mapsto \sum \lambda g(c).$$

Then, $\mathcal{I}(i) = (\mathcal{I}(i)_j, \varphi_\alpha)$ is the injective representation at vertex i.

Computing both the projective and injective representations for a quiver is not difficult, as often it relies on simply counting the number of paths between the vertices. Since we know that representations are isomorphic up to their linear maps, it often suffices to simply state the representation by presenting the relevant dimension vector. We also note that in the case of Q containing an orientated cycle there is some vertex i such that we will have an infinite-dimensional vector space in the representation $\mathcal{P}(i)$.

Example 2.15. Fix Q to be as follows;

$$\begin{tikzcd} 1 \arrow[rr] \arrow[dr] \arrow[rrrr, bend left] & & 3 \arrow[rr] & & 4 \\ & 2 \arrow[ur] & & & \end{tikzcd} \tag{2.2}$$

We will compute $\mathcal{S}$, $\mathcal{P}$, and $\mathcal{I}$. As mentioned above, computing $\mathcal{S}(i) \in \operatorname{rep} Q$ is trivial, e.g. we have

$$\mathcal{S}(2): \qquad \begin{tikzcd}
0 \arrow[rr] \arrow[dr] \arrow[rrrr, bend left] & & 0 \arrow[rr] & & 0 \\
& \Bbbk \arrow[ur] & & &
\end{tikzcd}$$

We now compute $\mathcal{P}(1) \in \operatorname{rep} Q$. The dimension of $\mathcal{P}(1)_1$ is just 1 as there is just the trivial path going from vertex $1 \to 1$. The dimension of $\mathcal{P}(1)_2$ is likewise just 1 as there is only one path from $1 \to 2$. The dimension of $\mathcal{P}(1)_3$ is 2 since there are two paths from $1 \to 3$; the first being $1 \to 3$ and the second being $1 \to 2 \to 3$. The dimension of $\mathcal{P}(1)_4$ is 3 since there are three paths from $1 \to 4$; the first $1 \to 4$, second $1 \to 3 \to 4$, and last $1 \to 2 \to 3 \to 4$. We see then that $\underline{\dim}\mathcal{P}(1) = (1, 1, 2, 3)$.

$$\mathcal{P}(1): \qquad \begin{tikzcd}
\Bbbk \arrow[rr] \arrow[dr] \arrow[rrrr, bend left] & & \Bbbk^2 \arrow[rr] & & \Bbbk^3 \\
& \Bbbk \arrow[ur] & & &
\end{tikzcd}$$

We now consider $\mathcal{I}(4) \in \operatorname{rep} Q$. The dimension of $\mathcal{I}(4)_1$ is 3 since there are three paths from $1 \to 4$, the ones described above for $\mathcal{P}(1)_4$. The dimension of $\mathcal{I}(4)_2$ is 1 since there is just the path $2 \to 3 \to 4$. Likewise, the dimensions of $\mathcal{I}(4)_3$ and $\mathcal{I}(4)_4$ are both just 1. So $\underline{\dim}\mathcal{I}(4) = (3, 1, 1, 1)$.

$$\mathcal{I}(4): \qquad \begin{tikzcd}
\Bbbk^3 \arrow[rr] \arrow[dr] \arrow[rrrr, bend left] & & \Bbbk \arrow[rr] & & \Bbbk \\
& \Bbbk \arrow[ur] & & &
\end{tikzcd}$$

We list all such representations of Q in Figure 2.1.

i	$\mathcal{S}(i)$	$\mathcal{P}(i)$	$\mathcal{I}(i)$
1	$\begin{tikzcd} \Bbbk \arrow[rr] \arrow[dr] \arrow[rrr, bend left] & & 0 \arrow[r] & 0 \\ & 0 \arrow[ur] & & \end{tikzcd}$	$\begin{tikzcd} \Bbbk \arrow[rr] \arrow[dr] \arrow[rrr, bend left] & & \Bbbk^2 \arrow[r] & \Bbbk^3 \\ & \Bbbk \arrow[ur] & & \end{tikzcd}$	$\begin{tikzcd} \Bbbk \arrow[rr] \arrow[dr] \arrow[rrr, bend left] & & 0 \arrow[r] & 0 \\ & 0 \arrow[ur] & & \end{tikzcd}$
2	$\begin{tikzcd} 0 \arrow[rr] \arrow[dr] \arrow[rrr, bend left] & & 0 \arrow[r] & 0 \\ & \Bbbk \arrow[ur] & & \end{tikzcd}$	$\begin{tikzcd} 0 \arrow[rr] \arrow[dr] \arrow[rrr, bend left] & & \Bbbk \arrow[r] & \Bbbk \\ & \Bbbk \arrow[ur] & & \end{tikzcd}$	$\begin{tikzcd} \Bbbk \arrow[rr] \arrow[dr] \arrow[rrr, bend left] & & 0 \arrow[r] & 0 \\ & \Bbbk \arrow[ur] & & \end{tikzcd}$
3	$\begin{tikzcd} 0 \arrow[rr] \arrow[dr] \arrow[rrr, bend left] & & \Bbbk \arrow[r] & 0 \\ & 0 \arrow[ur] & & \end{tikzcd}$	$\begin{tikzcd} 0 \arrow[rr] \arrow[dr] \arrow[rrr, bend left] & & \Bbbk \arrow[r] & \Bbbk \\ & 0 \arrow[ur] & & \end{tikzcd}$	$\begin{tikzcd} \Bbbk^2 \arrow[rr] \arrow[dr] \arrow[rrr, bend left] & & \Bbbk \arrow[r] & 0 \\ & \Bbbk \arrow[ur] & & \end{tikzcd}$
4	$\begin{tikzcd} 0 \arrow[rr] \arrow[dr] \arrow[rrr, bend left] & & 0 \arrow[r] & \Bbbk \\ & 0 \arrow[ur] & & \end{tikzcd}$	$\begin{tikzcd} 0 \arrow[rr] \arrow[dr] \arrow[rrr, bend left] & & 0 \arrow[r] & \Bbbk \\ & 0 \arrow[ur] & & \end{tikzcd}$	$\begin{tikzcd} \Bbbk^3 \arrow[rr] \arrow[dr] \arrow[rrr, bend left] & & \Bbbk \arrow[r] & \Bbbk \\ & \Bbbk \arrow[ur] & & \end{tikzcd}$

Figure 2.1: $\mathcal{S}$, $\mathcal{P}$, and $\mathcal{I}$ representations for Q as in (2.2).

Proposition 2.2. *Let Q be a quiver with no orientated cycles, then a vertex $i \in Q_0$ is a* sink *(resp.* source*) if and only if $\mathcal{S}(i) = \mathcal{P}(i)$ (resp. $\mathcal{S}(i) = \mathcal{I}(i)$).*

Proof. This follows almost immediately. Observe that $\mathcal{S}(i) = \mathcal{P}(i)$ if and only if the vertex i contains no paths from itself to another arrows, that is all arrows adjacent to i are facing in. Since all arrows point towards i we have that i is a sink. Likewise, $\mathcal{S}(i) = \mathcal{I}(i)$ if and only if the vertex i contains no paths from other arrows to itself, i.e. all arrows adjacent to i are facing out. It follows that i is a source then. $\square$

In the case of Example 2.15 we see that $\mathcal{S}(4) = \mathcal{P}(4)$, thus $i = 4$ is a sink, and $\mathcal{S}(1) = \mathcal{I}(1)$ so $i = 1$ is a source.

We now shift towards a series of results building off each other to end this section.

Proposition 2.3. *Let Q be a quiver with no orientated cycles, then the representations $\mathcal{S}(i), \mathcal{P}(i), \mathcal{I}(i) \in \operatorname{rep} Q$ are indecomposable for all vertices $i \in Q_0$.*

Proof. For $\mathcal{S}(i)$ this follows immediately since $\mathcal{S}(i)$ is simple.

Consider $\mathcal{P}(i) = (\mathcal{P}(i)_j, \varphi_\alpha)$, since we Q contains no orientated cycles we know $\mathcal{P}(i)_i = \Bbbk$. Suppose towards contradiction that $\mathcal{P}(i) = M \oplus N$ for $M, N \in \operatorname{rep} Q$, without loss of generality say $\mathcal{P}(i)_i = M_i \oplus N_i = \Bbbk \oplus 0$. Take $j \in Q_0$ to be a vertex such that $N_j \neq 0$, let $c = (i \mid \beta_1, \ldots, \beta_t \mid j)$ be a path which acts as a basis element of $\mathcal{P}(i)_j$. Let φ_c be the composition of the maps along the path c in $\mathcal{P}(i)$. It follows that since $\mathcal{P}(i) = M \oplus N$ the map $\varphi_c = \varphi_{\beta_t} \cdots \varphi_{\beta_1}$ also has the form of

$$\varphi_c : M_i \oplus 0 \to M_j \oplus N_j.$$

Which sends the (unique) basis e_i of M_i to an element $\varphi_c(e_i)$ of M_j. However from the definition of $\mathcal{P}(i)$ we know that $\varphi_c(e_i) = c$, which implies that every basis element c of $\mathcal{P}(i)_j$ lies in M_j which is a contradiction. Thus $\mathcal{P}(i)$ must be indecomposable.

The proof for $\mathcal{I}(i) = M \oplus N$ is similar, taking the same approach as above, we arrive at the contradiction that all basis elements of $\mathcal{I}(i)_j$ are contained in M_j contradicting N_j being non-zero. $\square$

We now define the last class of representations of Q.

Definition 2.16. Let $A = \bigoplus_{i \in Q_0} \mathcal{P}(i)$. Then a representation $F \in \operatorname{rep} Q$ is called *free* if $F \cong A \oplus \cdots \oplus A$.

Theorem 2.4. *A representation $M \in \operatorname{rep} Q$ is projective if and only if there is a free representation $F \in \operatorname{rep} Q$ such that M is isomorphic to a direct summand of F.*

Proof. ($\Rightarrow$) Suppose that M is projective with dimension vector $\underline{\dim}(M) = (d_i)_{i \in Q_0}$, with the standard projective resolution of M we see that there is a morphism $g : (d_i)\mathcal{P}(i) \to M$ inducing the short exact sequence:

$$0 \longrightarrow \ker g \xrightarrow{\text{incl}} \oplus d_i \mathcal{P}(i) \xrightarrow{g} M \longrightarrow 0 .$$

Since M is projective this sequence then splits and thus M is isomorphic to a direct summand of the $\mathcal{P}(i)$ as desired.

($\Leftarrow$) Suppose that $M \cong F$ for $F \in \operatorname{rep} Q$ free. By Krull-Schmidt we have that every direct summand of F is a direct sum of the $\mathcal{P}(i)$ and thus it is projective by Proposition 2.3. $\square$

Corollary 2.5. *Let $P \in \operatorname{rep} Q$ be a projective representation, then*

$$P \cong \mathcal{P}(i_1) \oplus \cdots \oplus \mathcal{P}(i_t),$$

where $i_1, \ldots, i_t$ are not necessarily distinct.

Lastly, we have the following result which will be used later in Section 2.3.

Theorem 2.6. *Let $M = (M_i, \varphi_\alpha) \in \operatorname{rep} Q$, then for any vertex $i \in Q_0$ there is an isomorphism of vector spaces:*

$$\operatorname{Hom}(\mathcal{P}(i), M) \cong M_i.$$

Proof. We construct an explicit map and show that this is an isomorphism. Let e_i denote the constant path (of length 0) at vertex i, then the space $\mathcal{P}(i)_i = \langle e_i \rangle$ has basis e_i. Let $\phi : \operatorname{Hom}(\mathcal{P}(i), M) \to M_i$ be defined as $\phi : (f_j)_{j \in Q_0} \mapsto f_i(e_i)$, where $f = (f_j)_{j \in Q_0} : \mathcal{P}(i) \to M$ is a morphism of quiver representations. The element $f_i \in f$ is a linear map from $\mathcal{P}(i)_i \to M_i$, showing that ϕ is well-defined as $e_i \in \mathcal{P}(i)_i$.

If $f, g \in \operatorname{Hom}(\mathcal{P}(i), M)$ are two morphisms of quiver representations then $\phi(f + g) = (f + g)_i(e_i) = f_i(e_i) + g_i(e_i) = \phi(f) + \phi(g)$. Furthermore, for $\lambda \in \Bbbk$ then $\phi(\lambda f) = (\lambda f)_i(e_i) = \lambda f_i(e_i) = \lambda \phi(f)$. So ϕ is a linear map.

We first show ϕ is injective. If $0 = \phi(f)$ then the linear map f_i sends the basis to zero and thus f_i is the zero map, we claim that for all $j \in Q_0$ then f_j is the zero map. To see this observe that the space $\mathcal{P}(i)_j$ has a basis consisting of all paths from $i \to j \in Q$, let $c = (i \mid \alpha_1, \dots, \alpha_t \mid j)$ be one such element. Consider the maps φ_c and φ'_c (these are the composition of the maps along the path c for representations $\mathcal{P}(i)$ and M respectively), it follows that $\varphi_c(e_i) = c$. Since f is a morphism we see that $f_j\varphi_c = \varphi'_c f_i$ but $f_i(e_i) = 0$ asserts that $f_i = 0$ and thus $f_j : c \mapsto 0$, our choice of c as a basis element of $\mathcal{P}(i)_j$ then shows that $f_j = 0$. Hence, ϕ is an injective map.

Let $m_i \in M_i$, we need to construct a morphism $f \in \mathrm{Hom}(\mathcal{P}(i), M)$ such that $f_i : e_i \mapsto m_i$. Fix the linear map f_i to satisfy this, and since e_i is the unique basis element of $\mathcal{P}(i)_i$ this defines the map f_i uniquely. We extend this map to a morphism f by following paths in Q. For any path $c = (i \mid\mid j)$ we say $f_j(c) = \varphi'_c(m_i)$. This defines each map f_j from a basis element of $\mathcal{P}(i)_j$, extended this map linearly to whole space $\mathcal{P}(i)_j$ results in the collection of maps f being a morphism. Hence $\phi(f) = m_i$ and ϕ is surjective. Thus ϕ is an isomorphism and the theorem holds. $\square$

2.2 The Path Algebra $\Bbbk Q$

Recall in Definition 2.11 we introduced paths of a quiver, as it turns out one can form a $\Bbbk$-algebra (see Definition 1.16) whose basis is the set of all paths in a fixed quiver. This section introduces this algebra formally and develops some mechanisms, setting up the context of Theorem 2.7.

Equipped with two paths $c = (i \mid \alpha_1, \dots, \alpha_t \mid j)$ and $c' = (j \mid \beta_1, \dots, \beta_s \mid k)$ we define the concatenation of paths $c \cdot c'$ as a new path

$$c \cdot c' = (i \mid \alpha_1, \dots, \alpha_t, \beta_1, \dots, \beta_s \mid k).$$

Definition 2.17. Let Q be a quiver, the *path algebra* of Q, $\Bbbk Q$, is the algebra with basis the set of all paths in Q and multiplication defined as

$$cc' = \begin{cases} c \cdot c' & t(c) = s(c') \\ 0 & \text{otherwise} \end{cases}$$

We explicitly show the associativity of $\Bbbk Q$. Consider $x, y, z \in \Bbbk Q$ then $x(yz) = (xy)z$ since

$$x(yz) = \begin{cases} x \cdot (yz) & t(x) = s(yz) \\ 0 & \text{otherwise} \end{cases} = \begin{cases} x \cdot y \cdot z & t(x) = s(yz) = s(y) \\ 0 & \text{otherwise} \end{cases}$$

and

$$(xy)z = \begin{cases} (xy) \cdot z & t(xy) = s(z) \\ 0 & \text{otherwise} \end{cases} = \begin{cases} x \cdot y \cdot z & t(xy) = t(y) = s(z) \\ 0 & \text{otherwise} \end{cases}$$

There is a direct decomposition of $\Bbbk Q$

$$\Bbbk Q = \Bbbk Q_0 \oplus \Bbbk Q_1 \oplus \cdots \oplus \Bbbk Q_\ell \oplus \cdots$$

where $\Bbbk Q_\ell$ is the subspace of $\Bbbk Q$ consisting of basis of all paths of length ℓ. Let c and c' be paths of lengths m and n respectively. From our definition cc' is either 0 or $cc' = c \cdot c'$, in the latter case we see that the length of cc' is obviously $m + n$. Thus, it follows that $\Bbbk Q$ is a $\mathbb{Z}$-graded $\Bbbk$-algebra.

The identity element $1 \in \Bbbk Q$ is defined as the sum of all constant paths, i.e. $1 = \sum_{i \in Q_0} e_i$. Furthermore, $e_i e_j = 0$ unless $i = j$, $e_i \alpha = 0$ unless $i = t(\alpha)$, and $\alpha e_i = 0$ unless $i = s(\alpha)$, thus the e_i are the *orthogonal idempotents* of $\Bbbk Q$.

Example 2.18. Fix the self-loop quiver L_1, i.e.;

$$\begin{tikzcd} 1 \arrow[loop right, "\alpha"] \end{tikzcd}$$

Which further generalizes to the r-loop quiver L_r consisting of 1 vertices and r self-loops:

$$\begin{tikzcd} 1 \arrow[loop above, "\alpha_2"] \arrow[loop right, "\alpha_1"] \arrow[loop below, "\alpha_r"] \arrow[loop left, "\cdots"] \end{tikzcd}$$

It is straightforward to see that the paths of L_1 are of the form $e_1, \alpha, \alpha{\cdot}\alpha, \alpha{\cdot}\alpha{\cdot}\alpha, \ldots$, alternatively, $e_1, \alpha, \alpha^2, \alpha^3, \ldots$ and thus $\Bbbk L_1 \cong \Bbbk[X]$. For the general L_r we see $\Bbbk L_r \cong \Bbbk[X_1, \ldots, X_r]$.

Consider M as a $\Bbbk Q$-module, even more broadly take M to be an A-module for some algebra A. The set of all *endomorphisms* $f : M \to M$, denoted by $\operatorname{End} M$, has a natural $\Bbbk$-vector space structure just as with Hom which further forms a $\Bbbk$-algebra under multiplication being defined as composition of endomorphisms.

Example 2.19. Fix Q to be the Kroenecker 2-quiver;

$$\begin{tikzcd} 1 \arrow[r, shift left, "\alpha"] \arrow[r, shift right, "\beta"'] & 2 \end{tikzcd} .$$

Then, the path algebra $\Bbbk Q$ is a $\Bbbk$-algebra of dimension 4, i.e $\Bbbk Q \cong \Bbbk^4$. And elements $x \in \Bbbk Q$ are expressed as $x = \lambda_1 e_1 + \lambda_2 e_2 + \lambda_3 \alpha + \lambda_4 \beta$, for $\lambda_i \in \Bbbk$. Define the representation $M \in \operatorname{rep} Q$

$$\begin{tikzcd} \Bbbk^2 \arrow[r, shift left, "T"] \arrow[r, shift right, "S"'] & \Bbbk^3 \end{tikzcd}$$

with maps T and S given as

$$T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \qquad S = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Then an endomorphism of M is a map $f : M \to M$ given as f_1 and f_2 where. It follows that for f to be a morphism we must have $f = (f_1, f_2) = (I_2, I_3)$, and thus we have $\operatorname{End} M \cong \mathbb{k}$.

We further define a new algebra from $\mathbb{k}Q$—the *opposite algebra*, denoted $\mathbb{k}Q^{\mathrm{op}}$. Let A be some $\mathbb{k}$-algebra then, one can define the opposite algebra A^{op} on the same underlying vector space structure as A, but by defining multiplication ab in A^{op} to be the same as ba in A. For a quiver Q, we form the *opposite quiver* Q^{op} by changing the orientations of arrows in Q, i.e. for $i \xrightarrow{\alpha} j \in Q_1^{\mathrm{op}}$ we have $j \xrightarrow{\alpha} i \in Q_1$. It should be immediate that $\mathbb{k}Q^{\mathrm{op}} = (\mathbb{k}Q)^{\mathrm{op}}$.

2.2.1 The Categories $\operatorname{rep} Q$ and $\operatorname{mod} \mathbb{k}Q$

Recall the material in Section 1.1.1, here we present a proof The proof of this theorem is adapted from its presentation in [CB92] and [Cum11].

Theorem 2.7. *For a fixed quiver Q, the category of finite-dimensional representations $\operatorname{rep} Q$ is equivalent to the category of finite-dimensional $\mathbb{k}Q$-modules $\operatorname{mod} \mathbb{k}Q$:*

$$\operatorname{rep} Q \cong \operatorname{mod} \mathbb{k}Q.$$

Proof. We prove the equivalence by constructing two functors F and G and show that they are inverses. Suppose Q is a finite quiver with n vertices, i.e. $n = |Q_0|$.

Let $F : \operatorname{rep} Q \to \operatorname{mod} \Bbbk Q$ such that for some representation $M \in \operatorname{rep} Q$ we have $F : M \mapsto V$ where $V := \oplus_{i \in Q_0} M_i$. It is easy to see that V is a graded $\Bbbk$-vector space. We first consider how F interacts with the vector spaces of $\operatorname{rep} Q$. Let

$$\iota_i : M_i \to V \qquad \pi_i : V \to M_i$$

be the canonical inclusion and projections, we define the module action for $v \in V$, $e_i, c = (i \mid \alpha_1, \ldots, \alpha_t \mid j) \in \Bbbk Q$ of $\Bbbk Q$ on V as follows;

(i) $v \cdot e_i = \iota_i \circ \pi_i(v)$,

(ii) $v \cdot c = v \cdot (i \mid \alpha_1, \ldots, \alpha_t \mid j) = \iota_j \circ (v_{\alpha_t} \cdots v_{\alpha_1}) \circ \pi_i(v)$.

Let $f : M \to N$ be a morphism of quiver representations M and N. From above, let $F(M) = V$ and $F(N) = W$, now consider the mapping

$$\psi : V \to W, \qquad \psi : (v_1, \ldots, v_n) \mapsto (f_1(v_1), \ldots, f_n(v_n)).$$

We now show that ψ is a $\Bbbk Q$ module homomorphism. Consider $v = (v_1, \ldots, v_n), v' = (v'_1, \ldots, v'_n) \in V$ then

$$\begin{aligned} \psi(v + v') = \psi(v_1 + v'_1, \ldots, v_n + v'_n) &= (f_1(v_1 + v'_1), \ldots, f_n(v_n + v'_n)) \\ &= (f_1(v_1), \ldots, f_n(v_n)) + (f_1(v'_1), \ldots, f_n(v'_n)) \\ &= \psi(v) + \psi(v') \end{aligned}$$

For $v = (v_1, \dots, v_n) \in V$ and $c = \alpha_1 \cdots \alpha_t \in \Bbbk Q$ we see that

$$\begin{aligned}
\psi(vc) &= \psi\left((v_1, \dots, v_n)(\alpha_1 \cdots \alpha_t)\right) \\
&= \psi\Big((0, \dots, 0, \underbrace{v_{\alpha_t} \cdots v_{\alpha_1}(v_{s(\alpha_1)})}_{t(\alpha_t)\text{th index}}, 0, \dots, 0)\Big) \\
&= \big(0, \dots, 0, f_{t(\alpha_t)}(v)(v_{\alpha_n} \cdots v_{\alpha_1}(v_{s(\alpha_1)})), 0, \dots, 0\big) && (1) \\
&= \big(0, \dots, 0, w_{\alpha_t} \cdots w_{\alpha_1}(f_{s(\alpha_1)}(v_{s(\alpha_1)})), 0, \dots, 0\big) && (2) \\
&= (f_1(v_1), \dots, f_n(v_n)) \cdot (\alpha_1 \cdots \alpha_t) \\
&= \psi\left(v_1, \dots, v_n\right) \cdot (\alpha_1 \cdots \alpha_t) = \psi(v) \cdot c
\end{aligned}$$

Where (1) to (2) occurs since the following diagram is commutative:

$$\begin{tikzcd}
M_{s(\alpha_1)} \arrow[r, "m_{\alpha_1}"] \arrow[d, "f_{s(\alpha_1)}"'] & M_{t(\alpha_1)} \arrow[r, "m_{\alpha_2}"] \arrow[d, "f_{t(\alpha_1)}"] & M_{t(\alpha_2)} \arrow[r] \arrow[d, "f_{t(\alpha_2)}"] & \cdots \arrow[r] & M_{t(\alpha_t)} \arrow[d, "f_{t(\alpha_t)}"] \\
N_{s(\alpha_1)} \arrow[r, "n_{\alpha_1}"] & N_{t(\alpha_1)} \arrow[r, "n_{\alpha_2}"] & N_{t(\alpha_2)} \arrow[r] & \cdots \arrow[r] & N_{t(\alpha_t)}
\end{tikzcd}$$

We now show that F is a functor, preserving identity and composition. Identity follows trivially since if f_i is the identity map for all vertices $i \in Q_0$ then $\psi(v_1, \dots, v_n) = (v_1, \dots, v_n)$. For composition let $f : L \to M$ and $g : M \to N$ be morphisms in $\operatorname{rep} Q$, by definition $f \circ g$ is another morphism $f \circ g : L \to N$, thus we need to show $F(f \circ g) = F(f) \circ F(g)$. Observe

$$\begin{aligned}
F(f \circ g)(l_1, \dots, l_n) &= (f_1 \circ g_1(l_1), \dots, f_n \circ g_n(l_n)) \\
&= F(f)\left(g_1(l_1), \dots, g_n(l_n)\right) = F(f) \circ F(g)(l_1, \dots, l_n).
\end{aligned}$$

It follows then that $F : \operatorname{rep} Q \to \operatorname{mod} \Bbbk Q$ is a functor.

Let $G : \operatorname{mod} \Bbbk Q \to \operatorname{rep} Q$ such that for some module $V \in \operatorname{mod} \Bbbk Q$ we have $G : V \mapsto M$ where $M = (Me_i, \varphi_\alpha)$ is defined as the collection $\Bbbk$-vector spaces $Me_i := \{ve_x \mid v \in V\}$ and $\varphi_\alpha : M_{s(\alpha)} \to M_{t(\alpha)}$ is defined as

$$\varphi_\alpha(ve_{s(\alpha)}) = (ve_{s(\alpha)}) \cdot \alpha = (m \cdot \alpha) = (m \cdot \alpha) \cdot e_{t(\alpha)} \in M_{t(\alpha)} = Ve_{t(\alpha)}.$$

Let $\phi : V \to W$ be a $\Bbbk Q$-module homomorphism, $G(V) = M$, and $G(W) = N$ consider the mapping

$$f : M \to N \qquad f_i : Ve_i \to We_i \qquad f_i : ve_i \mapsto \phi(v)e_i.$$

For f a morphism of representations of a quiver we need the following diagram to commute for all $\alpha \in Q_1$.

$$\begin{tikzcd}
M_{s(\alpha)} \arrow[r, "m_\alpha"] \arrow[d, "f_{s(\alpha)}"'] & M_{t(\alpha)} \arrow[d, "f_{t(\alpha)}"] \\
N_{s(\alpha)} \arrow[r, "n_\alpha"] & N_{t(\alpha)}
\end{tikzcd}$$

Let $ve_{s(\alpha)} \in M_{s(\alpha)}$, then

$$f_{t(\alpha)}\left(m_\alpha(ve_{s(\alpha)})\right) = f_{t(\alpha)}(v \cdot \alpha) = f_{t(\alpha)}\left((v \cdot \alpha) \cdot e_{t(\alpha)}\right) = f(v \cdot \alpha) \cdot e_{t(\alpha)} = f(v) \cdot \alpha$$

and

$$n_\alpha\left(f_{s(\alpha)}(ve_{s(\alpha)})\right) = n_\alpha\left(f(v) \cdot e_{s(\alpha)}\right) = \left(f(v) \cdot e_{s(\alpha)}\right) \cdot \alpha = f(v) \cdot \alpha.$$

So the diagram commutes and thus $G : \phi \mapsto f$ maps $\Bbbk Q$-module homomorphisms to quiver representation morphisms.

We now show that G is a functor. Identity follows trivially, let $\phi : U \to V$ and $\theta : V \to W$ be $\Bbbk Q$-module homomorphisms. By definition we have that $G(\phi) = f : G(U) \to G(V)$ where $f = (f_i)$ is the collection of $f_i : Ue_i \to Ve_i$ and $G(\theta) = g : G(V) \to G(W)$ and g has collection $g_i : Ve_i \to We_i$. We show that $G(\theta \circ \phi) = G(\theta) \circ G(\phi)$. For $i \in Q_0$ and $u \in U$ we have

$$G(\theta_i \circ \phi_i)(ue_i) = \theta \circ \phi(u)e_i = G(\theta_i)(\phi(u)e_i) = G(\theta_i) \circ G(\phi_i)(ue_i).$$

Thus $G : \operatorname{mod} \Bbbk Q \to \operatorname{rep} Q$ is a functor.

By construction, we see that $F \circ G = \mathrm{id}_{\operatorname{mod} \Bbbk Q}$ and $G \circ F = \mathrm{id}_{\operatorname{rep} Q}$, so F and G are inverses to each other thus the two categories are equivalent. □

Thus, from here on it suffices to identify representations of Q with right modules over $\Bbbk Q$, and the category $\operatorname{rep} Q$ with $\operatorname{mod} \Bbbk Q$.

2.2.2 Bound Quiver Algebras

Recall Example 2.18 where we saw the path algebra of the self-loop quiver to be isomorphic to $\Bbbk[X]$. This example indicates that the existence of an orientated cycle in a quiver implies that the path algebra is infinite-dimensional. Since we are interested in only finite-dimensional algebras in this thesis, this motivates us to define *bound quiver algebras*.

Since $\Bbbk Q$ is $\mathbb{Z}$-graded we can define the *arrow ideal* R_Q of $\Bbbk Q$ to be the two-sided ideal generated by all arrows in Q. That is,

$$R_Q = \Bbbk Q_1 \oplus \Bbbk Q_2 \oplus \cdots \oplus \Bbbk Q_\ell \oplus \cdots$$

where $\Bbbk Q_\ell$ is the subspace of $\Bbbk Q$ with basis elements paths of length ℓ. Then, the

ℓ-th power of R_Q can be decomposed as

$$R_Q^\ell = \bigoplus_{m \geq \ell} \Bbbk Q_m$$

with basis elements all paths of length greater or equal to ℓ.

Definition 2.20. Let I be a (two-sided) ideal of $\Bbbk Q$. Then I is an *admissible ideal* if there exists an integer $m \geq 2$ such that

$$R_Q^m \subset I \subset R_Q^2.$$

The initial condition that $R_Q^m \subset I$ implies that an admissible ideal contains all paths of length greater or equal to m, and it is easy to see from this that modding I from $\Bbbk Q$ results in a finite-dimensional algebra. The second condition of $I \subset R_Q^2$ ensures that we do not remove any arrows from our original quiver (since R_Q^2 contains all paths of length 2 or more). This then leads to the following.

Definition 2.21. If I is an admissible ideal of $\Bbbk Q$, then (Q, I) is a *bound quiver* and the quotient algebra $\Bbbk Q/I$ is a *bound quiver algebra.*

One way of constructing such a bound quiver would be through relations. A relation ρ is a composition of arrows in Q_1 which we define to be 0. E.g, for $Q = L_1$ we can define the relation $\rho = \alpha\alpha\alpha$ thus we have the set of all possible paths in Q to be $e_1, \alpha, \alpha\alpha$ so $\Bbbk Q \cong \Bbbk^3$.

Alternatively, for some admissible ideal I suppose $I = \langle \sigma_1, \ldots, \sigma_s \rangle$ then for every pair of vertices $x, y \in Q_0$ the element $e_x \sigma_i e_y$ is a linear combination of paths from x to y and hence have a relation. Therefore, we have the set of relations $R := \{e_x \sigma_i e_y \mid 1 \leq i \leq s; x, y \in Q_0\}$. That is, for any admissible ideal I, there is a set of relations R which generates I.

Example 2.22. Fix Q to be as follows;

$$\begin{tikzcd} 1 \arrow[loop left, "\alpha"] \arrow[r, "\beta"] & 2 \end{tikzcd}.$$

Consider the relations $\rho_1 = \alpha^2\beta$ and $\rho_2 = \alpha^4$ which generate the ideal $I = \langle \rho_1, \rho_2 \rangle$. We show that I is admissible.

Consider $m = 3$, then any path of length greater or equal to 4 must contain α^3 or $\alpha^2\beta$ as an initial path so we have $R_Q^3 \subset I$. The other desired relation, $I \subset R_Q^2$ follows immediately since the generators of I are of length three. Therefore I is an admissible ideal, and $\Bbbk Q/I \cong \Bbbk^3$.

Remark. Different relations on the same quiver may lead to isomorphic algebras.

We say a finite-dimensional algebra A is called *basic* if for every set of primitive orthogonal idempotents $\{e_1, \ldots, e_n\}$ such that $1 = \sum_i e_i$, we have

$$e_i A \cong e_j A \qquad \text{if and only if} \qquad i = j.$$

If A is an algebra that is not basic, then this motivates the following [ASS06, I.6].

Definition 2.23. Assume that A is a $\Bbbk$-algebra with a complete set $\{e_1, \ldots, e_n\}$ of primitive orthogonal idempotents. A *basic algebra associated to* A is the algebra

$$A^b = e_A A e_A,$$

where $e_A = e_{s_1} + \cdots + e_{s_t}$ are chosen such that $e_{s_i}A \cong e_{s_j}$ if and only if $i = j$ and each module $e_i A$ is isomorphic to one of the modules $e_{s_1}A, \ldots, e_{s_t}A$.

We also have the following result from [ASS06, II.3].

Theorem 2.8. *Let A be a basic finite-dimensional $\Bbbk$-algebra then there exists some admissible ideal I of $\Bbbk Q$ such that $A \cong \Bbbk Q/I$.*

The vertices of the quiver are in bijection with a set of primitive, orthogonal idempotents $\{e_1, \ldots, e_n\}$ with the standard property of $1 = \sum_i e_i$, and the number of arrows from $e_i \to e_j$ is given as

$$\dim \Big(e_i \left(\operatorname{rad} A / (\operatorname{rad} A)^2 \right) e_j \Big).$$

That is all to say—within representation theory, the study of finite-dimensional $\Bbbk$-algebras is reduced to the study of bound quiver algebras.

It follows almost immediately that in the case of bound quiver algebras we can extend Theorem 2.7 to the following.

Theorem 2.9. *For a fixed connected quiver Q and I an admissible ideal in $\Bbbk Q$, the category of finite-dimensional representations* $\operatorname{rep}(Q, I)$ *is equivalent to the category of finite-dimensional* $(\Bbbk Q/I)$*-modules* $\operatorname{mod}(\Bbbk Q/I)$*:*

$$\operatorname{rep}(Q, I) \cong \operatorname{mod}(\Bbbk Q/I).$$

We omit the proof as it follows the same structure in Theorem 2.7, with the necessary substitutions.

In the case of bounded quivers it follows that simple representations of Q will be the same as in (Q, I), i.e. $\mathcal{S}_Q(i) = \mathcal{S}_{(Q,I)}(i)$. However, for projective and injective representations, we must construct them differently in (Q, I). We end this section by defining indecomposable projective and injective representations over bounded quivers and providing an example similar in style to that of Example 2.15.

Definition 2.24. A *projective representation* $\mathcal{P}(i) \in \operatorname{rep}(Q, I)$ is such that the dimension at each vertex j is the number of residue classes $c + I$ of paths from i to j. If $j \xrightarrow{\alpha} l$ is an arrow in (Q, I) then $\varphi_\alpha : \mathcal{P}(i)_j \to \mathcal{P}(i)_l$ is defined by composing the arrow α to the paths from $i \to j$. That is the injective map f acts on basis elements as,

$$f : c + I \mapsto c\alpha + I.$$

Then, $\mathcal{P}(i) = (\mathcal{P}(i)_j, \varphi_\alpha)$ is the indecomposable projective representation at vertex i.

Definition 2.25. A *injective representation* $\mathcal{I}(i) \in \mathrm{rep}(Q, I)$ is such that the dimension at each vertex is the number of residue classes $c+I$ of paths from j to i. If $j \xrightarrow{\alpha} l$ is an arrow in (Q, I) then $\varphi_\alpha : \mathcal{I}(i)_j \to \mathcal{I}(i)_l$ is the map which removes α from $c + I$ if $c + I$ begins with α, otherwise it sends $c + I$ to 0. That is the surjective map g acts on basis elements as,

$$g : c + I \mapsto \begin{cases} c' + I & c = \alpha c' \\ 0 & \text{otherwise} \end{cases}.$$

Then, $\mathcal{I}(i) = (\mathcal{I}(i)_j, \varphi_\alpha)$ is the indecomposable injective representation at vertex i.

Example 2.26. Fix Q to be as follows;

$$\begin{tikzcd}
 & & & 2 \arrow[dl, "\beta"'] & \\
6 & 5 \arrow[l, "\zeta"'] & 4 \arrow[l, "\epsilon"'] & & 1 \arrow[ul, "\alpha"'] \arrow[dl, "\gamma"] \\
 & & & 3 \arrow[ul, "\delta"] &
\end{tikzcd} \tag{2.3}$$

With $\rho_1 = \alpha\beta - \gamma\delta$, $\rho_2 = \beta\epsilon$, and $\rho_3 = \delta\epsilon\zeta$. As an aside, consider $\rho_1' = \alpha\beta + \gamma\delta$, we can define two admissible ideals $I_1 = \langle \rho_1 \rangle$ and $I_2 = \langle \rho_1' \rangle$. For $\mathrm{char}(\Bbbk) \neq 2$ we have that $I_1 \neq I_2$ however the bound quivers $\Bbbk Q/I_1$ and $\Bbbk Q/I_2$ are isomorphic.

Now consider the set of relations defined as $R = \{\rho_1, \rho_2, \rho_3\}$ and then the ideal

$I = \langle R \rangle$. Computing the indecomposable projective representations for (Q, I) is then as straightforward as it was for unbounded quivers. We list all such $\mathcal{P}(i)$ in Figure 2.2.

2.3 Gabriel's Theorem

Motivated by the results of Theorem 2.7 and Theorem 2.9, we present the well-known Gabriel's theorem, originally presented in [Gab72] which sparked the interest and development of quiver theory. In 1973 Herstein, Gel'fand, and Ponomarev [BGP73] presented a cleaner proof of the statement, which also utilized the Coxeter functor—a topic we will discuss in Chapter 3. Our approach of the proof is taken from [Sch14], expository notes which follows the Herstein technique can be found in [Cum11] and [Hal21].

The statement itself is incredibly powerful and quite surprising, it connects finite algebras to the study of quivers, complementing Theorem 2.7 nicely. The theorem references *Dynkin diagrams* which are shown in Figure 2.3.

Theorem 2.10 (Gabriel 1972)**.** *Let Q be a connected quiver.*

1. *Then Q is of finite representation type if and only if the underlying Dynkin diagram, Δ_Q, of Q is of Dynkin type $\mathbb{A}$, $\mathbb{D}$, or $\mathbb{E}$.*

2. *If Q has underlying Dynkin type $\mathbb{A}$, $\mathbb{D}$, or $\mathbb{E}$ then the dimension vector induces a bijection from isoclasses of indecomposable representations of Q to the set of positive roots:*

$$\Psi : \operatorname{ind} Q \to \Phi_+ \qquad \Psi : M \mapsto \underline{\dim} M.$$

Prior to presenting the proof we need to develop some mechanisms and tools, we begin with a few necessary algebraic varieties. Let Q be a connected quiver without

i	$\mathcal{P}(i)$
1	$\begin{tikzcd} & & & \Bbbk \arrow[dl] & \\ 0 & 0 \arrow[l] & \Bbbk \arrow[l] & & \Bbbk \arrow[ul] \arrow[dl] \\ & & & \Bbbk \arrow[ul] & \end{tikzcd}$
2	$\begin{tikzcd} & & & \Bbbk \arrow[dl] & \\ 0 & 0 \arrow[l] & \Bbbk \arrow[l] & & 0 \arrow[ul] \arrow[dl] \\ & & & 0 \arrow[ul] & \end{tikzcd}$
3	$\begin{tikzcd} & & & \Bbbk \arrow[dl] & \\ 0 & 0 \arrow[l] & \Bbbk \arrow[l] & & 0 \arrow[ul] \arrow[dl] \\ & & & \Bbbk \arrow[ul] & \end{tikzcd}$
4	$\begin{tikzcd} & & & 0 \arrow[dl] & \\ \Bbbk & \Bbbk \arrow[l] & \Bbbk \arrow[l] & & 0 \arrow[ul] \arrow[dl] \\ & & & 0 \arrow[ul] & \end{tikzcd}$
5	$\begin{tikzcd} & & & 0 \arrow[dl] & \\ \Bbbk & \Bbbk \arrow[l] & 0 \arrow[l] & & 0 \arrow[ul] \arrow[dl] \\ & & & 0 \arrow[ul] & \end{tikzcd}$
6	$\begin{tikzcd} & & & 0 \arrow[dl] & \\ \Bbbk & 0 \arrow[l] & 0 \arrow[l] & & 0 \arrow[ul] \arrow[dl] \\ & & & 0 \arrow[ul] & \end{tikzcd}$

Figure 2.2: $\mathcal{P}$ representations for Q as in (2.3).

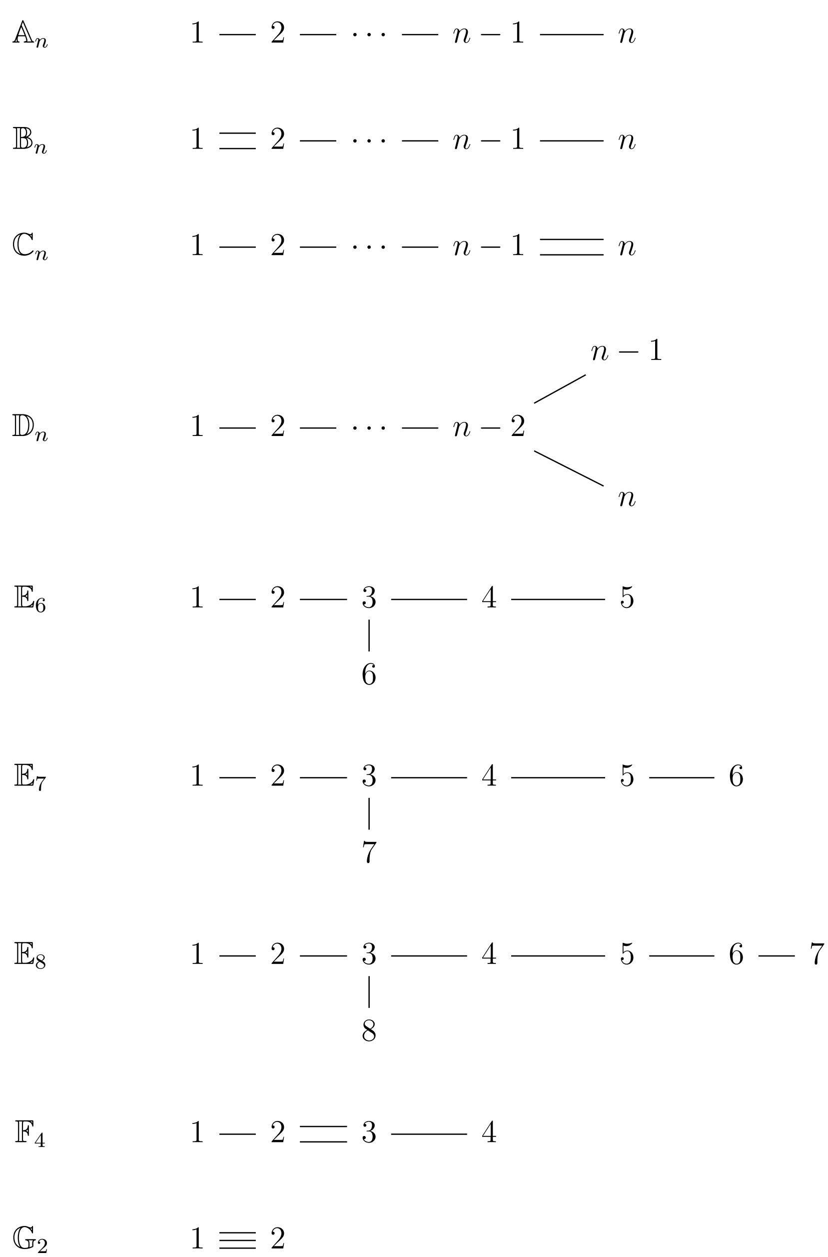


Figure 2.3: Dynkin diagrams.

orientated cycles, suppose $|Q_0| = n$, i.e., Q has n vertices. Fix some $\mathbf{d} = (d_i) \in \mathbb{Z}^n_{\geq 0}$ and define the space of all representations $M \in \operatorname{rep} Q$ with dimension vector $\mathbf{d}$ as $R_{\mathbf{d}}(Q) := \{M \in \operatorname{rep} Q \mid \underline{\dim} M = \mathbf{d}\}$. It follows that

$$R_{\mathbf{d}}(Q) = \bigoplus_{\alpha \in Q_1} \operatorname{Hom}_{\Bbbk}\left(\Bbbk^{d_{s(\alpha)}}, \Bbbk^{d_{t(\alpha)}}\right).$$

Moreover, $R_{\mathbf{d}}(Q)$ is a $\Bbbk$-vector space with dimension $\sum_\alpha d_{s(\alpha)} d_{t(\alpha)}$.

We then define the group

$$G_{\mathbf{d}} := \prod_{i \in Q_0} \mathrm{GL}_{d_i}(\Bbbk). \tag{2.4}$$

The group naturally acts on $R_{\mathbf{d}}(Q)$ via conjugation; if $g = (g_i) \in G_{\mathbf{d}}$, $M = (M_i, \varphi_\alpha) \in R_{\mathbf{d}}(Q)$, and $i \xrightarrow{\alpha} j \in Q_1$ then, $(g \cdot \varphi)_\alpha = g_j \varphi_\alpha g_i^{-1}$. We denote the orbit of $M \in R_{\mathbf{d}}(Q)$ under $G_{\mathbf{d}}$ by $\mathcal{O}_M := \{g \cdot M \mid g \in G_{\mathbf{d}}\}$. The following lemma shows that this orbit is in fact the isoclass of the representation M.

Lemma 2.11. *The orbit $\mathcal{O}_M$ is the isoclass of the representation M, that is,*

$$\mathcal{O}_M = \{M' \in \operatorname{rep} Q \mid M \cong M'\}.$$

Proof. Suppose that $M = (M_i, \varphi_\alpha)$ and $M' = (M'_i, \varphi'_\alpha)$ are in the same orbit, then there exists some $g \in G_{\mathbf{d}}$ such that $g \cdot M = M'$. That is, for each arrow $i \xrightarrow{\alpha} j$ in Q the following diagram commutes:

$$\begin{tikzcd} M_i \arrow[r, "\varphi_\alpha"] \arrow[d, "g_i"'] & M_j \arrow[d, "g_j"] \\ M'_i \arrow[r, "\varphi'_\alpha"'] & M'_j \end{tikzcd}$$

Therefore, g is a morphism of representations, moreover since each g_i is an element of $\mathrm{GL}_{d_i}(k)$ we have that it is invertible and thus an isomorphism. That is, $M \cong M'$.

It follows immediately that if $M \cong M'$ then there is a $g \in G_{\mathbf{d}}$ such that $g \cdot M = g(M) = M'$. $\square$

The *stabilizer* $\operatorname{Stab} M = \{g \in G_{\mathbf{d}} \mid g \cdot M = M\}$ corresponds with the automorphism group $\operatorname{Aut} M$ of the representation M. For any $\mathbf{d} \in \mathbb{Z}^n$ such that a representation M has dimension vector $\mathbf{d}$ we have the following equality of variety dimensions;

$$\dim \mathcal{O}_M = \dim G_{\mathbf{d}} - \dim \operatorname{Aut} M. \tag{2.5}$$

This follows from the natural bijection $(G_{\mathbf{d}}/\operatorname{Stab} M) \to \mathcal{O}_M$ defined by $\bar{g} \mapsto g \cdot M$, thus $\dim \mathcal{O}_M = \dim(G_{\mathbf{d}}/\operatorname{Stab} M)$ and of course $\dim(G_{\mathbf{d}}/\operatorname{Stab} M) = \dim G_{\mathbf{d}} - \dim \operatorname{Stab} M = \dim G_{\mathbf{d}} - \dim \operatorname{Aut} M$.

Another fact we can observe is that there is at most one orbit of O of codimension zero in $R_{\mathbf{d}}(Q)$, this follows since the algebraic variety $R_{\mathbf{d}}(Q)$ is irreducible and an orbit with codimension zero in $R_{\mathbf{d}}(Q)$ is open. See Appendix A.2 for further geometric properties of representations of quivers.

2.3.1 Tits Quadratic Form of a Quiver

We define the associated quadratic form of a quiver (sometimes referred to in literature as the Tits form or Tits quadratic form), building up to a key step in the proof of Gabriel's theorem (Theorem 2.14).

Definition 2.27. The *quadratic form* $q : \mathbb{Z}^n \to \mathbb{Z}$ of Q is defined as

$$q(x_1, \dots, x_n) = \sum_{i \in Q_0} x_i^2 - \sum_{\alpha \in Q_1} x_{s(\alpha)} x_{t(\alpha)}.$$

Note that q has no dependency on α-orientation, i.e., q associates to the underlying Dynkin diagram Δ_Q of Q.

Example 2.28. Let Q be the quiver from Example 2.6, $1 \longrightarrow 2 \longleftarrow 3$, it's corresponding quadratic form is then

$$q(x_1, x_2, x_3) = x_1^2 + x_2^2 + x_3^2 - x_1x_2 - x_2x_3.$$

It follows that the value of a quadratic form relies only on the dimension vector of a representation and not the actual representation itself. That is, q is constant in the space $R_{\mathbf{d}}(Q)$. The following proposition interprets q in the language of representation theory.

Lemma 2.12. *For any representation $M \in \operatorname{rep} Q$ with $\underline{\dim} M = \mathbf{d}$, we have*

$$q(\mathbf{d}) = \dim \operatorname{Hom}(M, M) - \dim \operatorname{Ext}^1(M, M).$$

Proof. Consider the standard projective resolution,

$$0 \longrightarrow \bigoplus_{\alpha \in Q_1} d_{s(\alpha)} \mathcal{P}(t(\alpha)) \xrightarrow{f} \bigoplus_{i \in Q_0} d_i \mathcal{P}(i) \xrightarrow{g} M \longrightarrow 0 \ .$$

Applying the $\operatorname{Hom}(-, M)$ functor then yields the exact sequence

$$\begin{tikzcd} 0 \arrow[r] & \operatorname{Hom}(M, M) \arrow[r] & \bigoplus_{i \in Q_0} d_i \operatorname{Hom}(\mathcal{P}(i), M) \arrow[dll, rounded corners] \\ \bigoplus_{\alpha \in Q_1} d_{s(\alpha)} \operatorname{Hom}(\mathcal{P}(t(\alpha)), M) \arrow[r] & \operatorname{Ext}^1(M, M) \arrow[r] & 0 \end{tikzcd}$$

since each $\mathcal{P}(i)$ is projective the last term of the sequence is zero. Then we conclude that

$$\sum_{i \in Q_0} d_i \dim \mathrm{Hom}(\mathcal{P}(i), M) - \sum_{\alpha \in Q_1} d_{s(\alpha)} \dim \mathrm{Hom}(\mathcal{P}(t(\alpha)), M) = \sum_{i \in Q_0} d_i^2 - \sum_{\alpha \in Q_1} d_{s(\alpha)} d_{t(\alpha)}$$

which is in fact $q(\mathbf{d})$, since we have the isomorphism $\mathrm{Hom}(\mathcal{P}(i), M) \cong M_i$ from Theorem 2.6. Hence,

$$\dim \mathrm{Hom}(M, M) - \dim \mathrm{Ext}^1(M, M) = \sum_{i \in Q_0} d_i^2 - \sum_{\alpha \in Q_1} d_{s(\alpha)} d_{t(\alpha)} = q(\mathbf{d})$$

as desired. □

Recall the simply laced Dynkin diagrams in Figure 2.3, in Figure 2.4 we present the Affine Dynkin diagrams sometimes referred to as Euclidean diagrams or extended Dynkin diagrams. These are constructed by adding one vertex to their corresponding Dynkin diagram and some amount of edges such that the diagram itself is not Dynkin, but by removing one vertex the resulting diagram is the union of Dynkin diagrams.

For a given quadratic form q, we say q is *positive definite* (resp. *positive semi-definite*) if $q(\mathbf{x}) > 0$ (resp. $q(\mathbf{x}) \geq 0$) for all non-zero $\mathbf{x} \in \mathbb{Z}^n$.

Lemma 2.13. *Let Q be a connected quiver with quadratic form q and $\mathbf{d} \in \mathbb{Z}^n \setminus \{0\}$ such that $(\mathbf{d}, \mathbf{x}) = 0$ for all $\mathbf{x} \in \mathbb{Z}^n$. Then,*

1. *q is positive semi-definite.*

2. *For all i, $d_i \neq 0$.*

3. *$q(\mathbf{x}) = 0$ if and only if $\mathbf{x} = c\mathbf{d}$ where $c = a/b \in \mathbb{Q}$.*

Proof. Let n_{ij} denote the number of *edges* between i and j in Q (note that this is the number of arrows $i \to j \in Q$ and $j \to i \in Q$). We also assume a standard

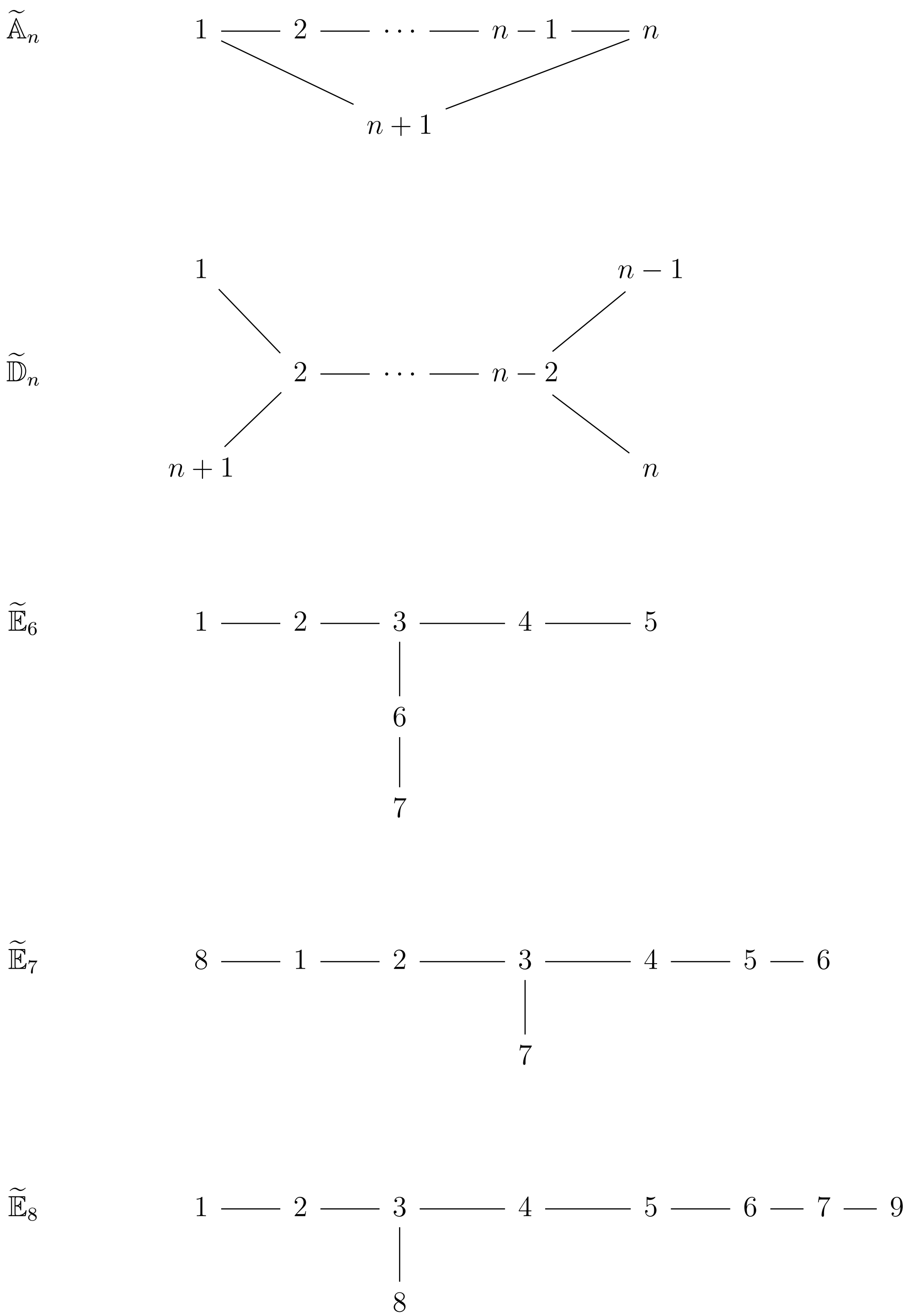


Figure 2.4: Affine Dynkin diagrams $\widetilde{\mathbb{A}}_n$, $\widetilde{\mathbb{D}}_n$, and $\widetilde{\mathbb{E}}_{6,7,8}$.

enumeration of vertices, i.e., $1, \ldots, n$. Then we have

$$q(\mathbf{x}) = \sum_{i=1}^{n} x_i^2 - \sum_{i=1}^{n} \sum_{j<i} n_{ij} x_i x_j \qquad \text{and} \qquad (\mathbf{x}, \mathbf{y}) = \sum_{i=1}^{n} 2x_i y_i - \sum_{i=1}^{n} \sum_{j \neq i} n_{ij} x_i y_j. \tag{1}$$

Suppose $\mathbf{d} \in \mathbb{Z}^n \setminus \{0\}$ and $(\mathbf{d}, \mathbf{x}) = 0$ for all non-zero $\mathbf{x} \in \mathbb{Z}^n$, let $\mathbf{e}_i$ be the ith standard basis vector of $\mathbb{Z}^n$. Then, $0 = (\mathbf{d}, \mathbf{e}_i) = 2d_i - \sum_{j \neq i} n_{ij} d_j$ hence;

$$d_i = \sum_{j<i} n_{ij} d_j. \tag{2}$$

Since $n_{ij} \geq 0$ we have that there is some vertex i such that $d_i = 0$, and for all its neighbors j we have $d_j = 0$. As Q is connected we have that this implies that $d_j = 0$ for all $j \in Q_0$, which contradicts our assumptions. This proves statement 2.

Let $\mathbf{x} \in \mathbb{Z}^n$, by (2) we have

$$\begin{aligned}
\sum_{i=1}^{n} x_i^2 &= \sum_{i} \frac{x_i^2}{d_i} \sum_{j<i} n_{ij} d_j = \sum_{i} \sum_{j<i} n_{ij} d_j \frac{x_i^2}{d_i} \\
&= \sum_{i} \sum_{j \neq i} \frac{n_{ij} d_j}{2} \frac{x_i^2}{d_i} &(3)\\
&= \sum_{i} \sum_{j<i} \left(\frac{n_{ij} d_j}{2} \frac{x_i^2}{d_i} + \frac{n_{ij} d_i}{2} \frac{x_j^2}{d_j} \right) = \sum_{i} \sum_{j<i} \frac{n_{ij} d_i d_j}{2} \left(\frac{x_i^2}{d_i^2} + \frac{x_j^2}{d_j^2} \right) &(4)
\end{aligned}$$

where (3) holds since the change from pairs (i, j) such that $j < i$ to pairs (i, j) such that $j \neq i$ is accounted for by dividing by 2. Then (4) holds since changing back from $j \neq i$ to $j < i$ we add the second summand. Combining (1) with (4) we see that

$$\begin{aligned}
q(\mathbf{x}) &= \sum_{i=1}^{n} x_i^2 - \sum_{i=1}^{n} \sum_{j<i} n_{ij} x_i x_j \\
&= \sum_{i} \sum_{j<i} \frac{n_{ij} d_i d_j}{2} \left(\frac{x_i^2}{d_i^2} + \frac{x_j^2}{d_j^2} - 2\frac{x_i x_j}{d_i d_j} \right) = \sum_{i} \sum_{j<i} \frac{n_{ij} d_i d_j}{2} \left(\frac{x_i}{d_i} - \frac{x_j}{d_j} \right)^2 &(5)
\end{aligned}$$

This immediately asserts statement 1. since $q(\mathbf{x}) \geq 0$ as $d_i, d_j > 0$ and $n_{ij} \geq 0$. Furthermore, the last equality in (5) shows that $q(\mathbf{x}) = 0$ if and only if $\frac{x_i}{d_i} = \frac{x_j}{d_j}$ for all vertices $i, j \in Q_0$, which shows statement 3. $\square$

The following is the main result of this section and connects Dynkin diagrams to the corresponding quadratic form.

Theorem 2.14. *Let Q be a connected quiver with quadratic form q and Δ_Q the underlying diagram of Q, then*

1. *q is positive definite if and only if Δ_Q is of Dynkin type $\mathbb{A}_n$, $\mathbb{D}_n$, or $\mathbb{E}_{6,7,8}$.*
2. *q is positive semi-definite if and only if Δ_Q is of Affine Dynkin type or Dynkin type $\mathbb{A}_n$, $\mathbb{D}_n$, or $\mathbb{E}_{6,7,8}$.*

Proof. We begin by showing that q is positive semi-definite if Δ_Q is of Affine Dynkin type, it suffices to find a dimension vector δ such that $(\delta, \mathbf{x}) = 0$ by Lemma 2.13, which are listed in Figure 2.5. It is easy to verify the desired condition holds.

Suppose q is positive semi-definite and Δ_Q is not simply laced Dynkin or Affine Dynkin. Then Q contains a subquiver Q' with $\Delta_{Q'}$ of Affine Dynkin type and q' as its quadratic form. Let δ be as in Figure 2.5. If $Q_0 = Q'_0$ then Q contains more arrows than Q' and then $q'(\delta) = 0 > q(\delta)$ a contradiction. If Q contains more vertices than Q', fix a vertex i_0 in Q which is connected by an arrow to a vertex j_0 in Q'. Define $\mathbf{x} = 2\delta$, then $x_{i_0} = 1$ and $x_j = 0$ for all other vertices $j \in Q_0$. So $q(\mathbf{x}) \geq q'(2\delta) + 1 - 2\delta_{j_0} = 1 - 2\delta_{j_0} < 0$, a contradiction. It then follows that if q is positive definite then Δ_Q must be simply laced Dynkin since for each Affine Dynkin diagram we have δ such that $q(\delta) = 0$. Hence, 2. is proven. $\square$

Along with quadratic forms, we have the notion of roots. Let $\mathbf{x} \in \mathbb{Z}^n \setminus \{0\}$, if $q(\mathbf{x}) = 1$ then $\mathbf{x}$ is a *real root*, if $q(\mathbf{x}) = 0$ then $\mathbf{x}$ is an *imaginary root*. We denote the

Δ_Q	δ
$\widetilde{\mathbb{A}}_n$	$(1, 1, \ldots, 1, 1)$
$\widetilde{\mathbb{D}}_n$	$(1, 2, 2, \ldots, 2, 1, 1, 1)$
$\widetilde{\mathbb{E}}_6$	$(1, 2, 3, 2, 1, 2, 1)$
$\widetilde{\mathbb{E}}_7$	$(2, 3, 4, 3, 2, 1, 2, 1)$
$\widetilde{\mathbb{E}}_8$	$(2, 4, 6, 5, 4, 3, 2, 3, 1)$

Figure 2.5: Vectors δ such that $(\delta, \mathbf{x}) = 0$ for all $\mathbf{x} \in \mathbb{Z}^n$, for Δ_Q Affine Dynkin.

set of all roots by Φ, and say $\mathbf{x}$ is *positive* (resp. *negative*) if $x_i \geq 0$ (resp. $x_i \leq 0$) for all $x_i \in \mathbf{x}$, the set of positive real roots is Φ_+ and the set of negative real roots is Φ_-.

2.3.2 Proof of Gabriel's Theorem

The following proposition relates the dimension of an orbit to the quadratic form.

Proposition 2.15. *Let Q be a connected quiver and $M \in \operatorname{rep} Q$ such that $\underline{\dim} M = \mathbf{d}$, then*

$$\operatorname{codim} \mathcal{O}_M = \dim \operatorname{End} M - q(\mathbf{d}) = \dim \operatorname{Ext}^1(M, M).$$

Proof. Observe,

$$\begin{aligned}
\operatorname{codim} \mathcal{O}_M &= \dim R_{\mathbf{d}}(Q) - \dim \mathcal{O}_M \\
&= \dim R_{\mathbf{d}}(Q) - (\dim G_{\mathbf{d}} - \dim \operatorname{Aut} M) && (2) \\
&= \dim R_{\mathbf{d}}(Q) - \sum_{i \in Q_0} d_i^2 + \dim \operatorname{End} M && (3) \\
&= \sum_{\alpha \in Q_0} d_{s(\alpha)} d_{t(\alpha} - \sum_{i \in Q_0} d_i^2 + \dim \operatorname{End} M
\end{aligned}$$

Where (2) follows from (2.5) and (3) follows from $\operatorname{Aut} M$ being an open subgroup of $\operatorname{End} M$. The second equality then follows from Lemma 2.12. $\square$

Suppose $\mathbf{d} \in \mathbb{Z}^n$ is such that $q(\mathbf{d}) \leq 0$ and $M \in \operatorname{rep} Q$ where $\underline{\dim} M = \mathbf{d}$. Then we see that $\operatorname{codim} \mathcal{O}_M \geq \dim \operatorname{End} M \geq 1$ therefore the dimension of $\operatorname{End} M$ is strictly greater than the dimension of any orbit $\mathcal{O}_M$. Hence the number of orbits is infinite, it follows then that there are infinitely many isoclasses of representations with dimension vector $\mathbf{d}$.

We may now prove Gabriel's theorem. The theorem is stated in two parts, we first present a proof of part two and then the proof of part one follows nicely.

Theorem 2.16. *Let Q be a connected quiver.*

2. *If Q has underlying Dynkin type $\mathbb{A}$, $\mathbb{D}$, or $\mathbb{E}$ then the dimension vector induces a bijection from isoclasses of indecomposable representations of Q to the set of positive roots:*

$$\Psi : \operatorname{ind} Q \to \Phi_+ \qquad \Psi : M \mapsto \underline{\dim} M.$$

Proof. To prove our statement we show that Ψ is well-defined, then that Ψ is injective, and lastly that Ψ is surjective.

Let M be an indecomposable representation of Q, we need to show that $q(\underline{\dim} M) = 1$, of which it suffices to show $\operatorname{End} M \cong \mathbb{k}$ and $\dim \operatorname{Ext}^1(M, M) = 0$. We first show that $\operatorname{End} M \cong \mathbb{k}$, we proceed by induction on the dimension of M. If M is a simple representation, then it follows immediately. Suppose M has dimension strictly greater than 1, since M is indecomposable this implies that for all $f \in \operatorname{End} M$, $f = \lambda 1_M + g$ where $\lambda \in k$ and $g \in \operatorname{End} M$ is a nilpotent endomorphism. Since g is nilpotent, without loss of generality, we assume that $g^2 = 0$, moreover we choose g such that $\dim(\operatorname{im} g)$ is minimal. Then, $\operatorname{im} g \subset \ker g$ therefore there exists some indecomposable subrepresentation L such that $\operatorname{im} g \cap L$ is non-zero.

Let $\pi : \ker g \to L$ be the canonical projection and i the non-zero morphism given by the incl : $\operatorname{im} g \to \ker g$ and π. That is,

$$\begin{tikzcd} \operatorname{im} g \arrow[r, "\text{incl}"] \arrow[rr, bend right, "i"'] & \ker g \arrow[r, "\pi"] & L \end{tikzcd}$$

This implies the composition $M \to \operatorname{im} g \to L \to M$ is a non-zero endomorphism whose square is zero. Then, the image is $i(\operatorname{im} g)$ and since g is taken to be minimal we have that $\dim i(img) \geq \dim(\operatorname{im} g)$ and thus i is injective. So the short exact sequence

$$\begin{tikzcd} 0 \arrow[r] & \operatorname{im} g \arrow[r, "i"] & L \arrow[r] & \operatorname{coker} i \arrow[r] & 0 \end{tikzcd}$$

can have the $\operatorname{Hom}(-, L)$ functor applied to it, resulting in the following surjective morphism

$$\begin{tikzcd} \operatorname{Ext}^1(L, L) \arrow[r] & \operatorname{Ext}^1(\operatorname{im} g, L) \arrow[r] & 0 \end{tikzcd} .$$

Then, by induction, $\dim \operatorname{Hom}(L, L) = 1$, and q is positive definite thus the dimension of $\operatorname{Ext}^1(L, L)$ is zero so the above equation shows that $\operatorname{Ext}^1(\operatorname{im} g, L) = 0$.

Consider the commutative diagram, whose rows are exact, and the bottom row is a push out of the top row along the morphism π.

$$\begin{tikzcd} 0 \arrow[r] & \ker g \arrow[r, "u"] \arrow[d, "\pi"'] & M \arrow[r, "g"] \arrow[d, "j_2"'] & \operatorname{im} g \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & L \arrow[r, "j_1"'] & X \arrow[r] & \operatorname{im} g \arrow[r] & 0 \end{tikzcd}$$

Since $\operatorname{Ext}^1(\operatorname{im} g, L) = 0$ this implies that the bottom row splits so there exists some morphism $h : X \to L$ such that $hj_1 = 1_L$. Let $\nu : L \to \ker g$ be the inclusion of the direct summand, so $\pi\nu = 1_L$. We then construct $hj_2 : M \to L$ and $u\nu : L \to M$ such that $hj_2u\nu = hj_1\pi\nu = 1_L 1_L = 1_L$ and thus L is a direct summand of M. Thus, M is indecomposable so L must be either 0 or M. However, $L \neq 0$ since $\operatorname{im} g \cap L$ is non-zero and $L \neq M$ since $L \subset \ker g$ and $g \neq 0$. Therefore we arrive at a contradiction and $\dim \operatorname{End}(M) = 1$, q is positive definite, $\dim \operatorname{Ext}^1(M, M) = 0$, and $q(\underline{\dim} M) = 1$. Hence $\underline{\dim} M$ is a positive root and Ψ is well-defined.

We now show that Ψ is injective. Let $M, M' \in \operatorname{rep} Q$ such that they are both indecomposable and $\underline{\dim} M = \underline{\dim} M'$. We know that for Dynkin types $\mathbb{A}$, $\mathbb{D}$, and $\mathbb{E}$ the indecomposable representations have no self-extensions. Therefore, the orbits $\mathcal{O}_M$ and $\mathcal{O}_{M'}$ both have codimension zero, which occurs when $M \cong M'$. This shows that Ψ is injective.

Now we show that Ψ is surjective. Let Q be of Dynkin type $\mathbb{A}$, $\mathbb{D}$, or $\mathbb{E}$, $\mathbf{d}$ a positive root, $M \in \operatorname{rep} Q$ such that $\underline{\dim} M = \mathbf{d}$ and $\mathcal{O}_M$ of maximum dimension in $R_{\mathbf{d}}(Q)$. We need to show that M is indecomposable. Let $M = M_1 \oplus M_2$, we will first show that $\operatorname{Ext}^1(M_1, M_2) = \operatorname{Ext}^1(M_2, M_1) = 0$. Suppose that $\operatorname{Ext}^1(M_1, M_2) \neq 0$, then this implies that there exists a non-split short exact sequence of the form

$$0 \longrightarrow M_2 \longrightarrow E \longrightarrow M_1 \longrightarrow 0$$

here $\underline{\dim} E = \underline{\dim} M$. Then a previous result implies that $\dim \mathcal{O}_M < \dim \mathcal{O}_E$, a contradiction of the maximality of $\mathcal{O}_M$. Thus, $\operatorname{Ext}^1(M_1, M_2) = 0$ and by symmetry we see that $\operatorname{Ext}^1(M_2, M_1) = 0$. Since $q(\mathbf{d}) = \dim \operatorname{Hom}(M, M) - \dim \operatorname{Ext}^1(M, M)$, we see that

$$1 = q(\mathbf{d}) = \dim \operatorname{Hom}(M_1 \oplus M_2, M_1 \oplus M_2) \geq 2$$

and arrive at a contradiction. Thus M is indecomposable, $\Psi(M) = \mathbf{d}$, and Ψ is surjective. $\square$

We can now prove Theorem 2.10 part 1.

Theorem 2.17. *Let Q be a connected quiver.*

1. *Then Q is of finite representation type if and only if the underlying Dynkin diagram, Δ_Q, of Q is of Dynkin type $\mathbb{A}$, $\mathbb{D}$, or $\mathbb{E}$.*

Proof. Suppose Q is not of Dynkin type $\mathbb{A}$, $\mathbb{D}$, or $\mathbb{E}$, then there exists some $\mathbf{d} \neq 0$ such that $q(\mathbf{d}) \leq 0$, so by the Corollary there are infinitely many isoclasses of representations with dimension vector $\mathbf{d}$. Each representation is a finite direct sum of indecomposable representations, therefore the number of isoclasses of indecomposable representations is infinite, thus the proof is complete. $\square$

We end this chapter by mentioning another well-established result in early quiver theory. In 1982 Kac extended the correspondence of indecomposable representations of a quiver to Kac-Moody Lie algebras [Kac82]. We present his theorem here, but omit a proof. An expository note by Lennen [Len19] details a proof of a weaker statement.

Theorem 2.18 (Kac 1982)**.** *Let Q be a quiver.*

1. *There exists an indecomposable representation of dimension $\mathbf{d}$ if and only if $\mathbf{d} \in \Phi_+$.*

2. *If $\mathbf{d} \in \Phi_+$ is real, then there exists a unique indecomposable representation of dimension $\mathbf{d}$.*

3. *If $\mathbf{d} \in \Phi_+$ is imaginary, then there are infinitely many indecomposable representations of dimension $\mathbf{d}$.*

Chapter 3: Auslander-Reiten Theory

As seen in the preceding chapter, quiver theory provides a convenient way of visualizing finite-dimensional algebras and their modules. Despite this elegant presentation, we are at an impasse since we cannot compute all indecomposable modules (we have simple, projective, and injective) and furthermore we have not defined the "irreducible" morphisms which connect them. Initially introduced in [Aus74], [AR75], and [AR77] Auslander-Reiten provided a theory introducing *almost split sequences* resulting in the following main theorem.

Theorem 3.1 (Auslander-Reiten, 1976)**.** *Let A be a finite dimensional $\Bbbk$-algebra and N be a finite-dimensional nonprojective indecomposable A-module. Then there exists a nonsplit short exact sequence*

$$0 \longrightarrow L \xrightarrow{f} M \xrightarrow{g} N \longrightarrow 0$$

in $\operatorname{mod} A$ *such that;*

1. *L is noninjective indecomposable,*
2. *if $u : L \to U$ is not a section then there exists some $u' : M \to U$ such that $u = u'f$, and*
3. *if $v : V \to N$ is not a retraction then there exists some $v' : V \to M$ such that $v = gv'$.*

Moreover, the sequence is uniquely determined up to isomorphism.

Dually, for any noninjective indecomposable A-module L, there is the same sequence where N is nonprojective indecomposable and 2. and 3. both hold. This particular sequence is called an *almost split sequence*, and allows us to study the short exact sequences of representations.

In this chapter we introduce the Auslander-Reiten quiver (AR-quiver), a quiver which approximates $\operatorname{rep} Q$ (or $\operatorname{mod} A$) and as we will see when Δ_Q is Dynkin type $\mathbb{A}$, $\mathbb{D}$, or $\mathbb{E}$ provides a full encoding of the category $\operatorname{rep} Q$. We then detail two methods of constructing the AR-quiver, and of course provide many examples. From the results in Chapter 2 (Theorem 2.7 and Theorem 2.9) we may think either in terms of representations of a quiver or modules of a finite $\Bbbk$-algebra.

3.1 Auslander-Reiten Quiver Γ_Q

As mentioned previously the AR-quiver provides a first approximation of the category $\operatorname{mod} A$ for some $\Bbbk$-algebra A. Formally we have the definitions.

Definition 3.1. A morphism $f : M \to N$ of representations is said to be *irreducible* if;

i. f is not a section or a retraction, and

ii. if $f = f_1 f_2$ then either f_1 is a retraction or f_2 is a section.

$$\begin{tikzcd} M \arrow[rr, "f"] \arrow[dr, "f_2"'] & & N \\ & L \arrow[ur, "f_1"'] & \end{tikzcd}$$

Giving further context to the AR-quiver defined below.

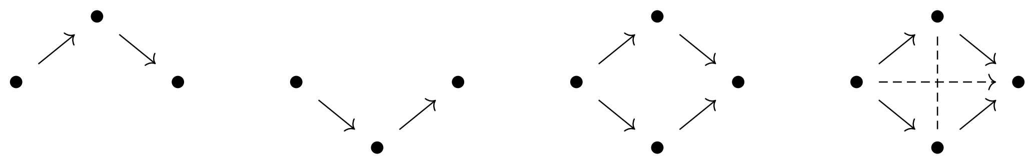


Figure 3.1: The four meshes within an AR-quiver.

Definition 3.2. Let A be a connected finite $\Bbbk$-algebra, then the *AR-quiver* Γ_A is defined as

- the vertices of Γ_A are the isoclasses $[M]$ of modules in $\operatorname{mod} A$,
- the arrows $[M] \to [N] \in \Gamma_A$ are the irreducible morphisms.

We detail the last key element of the AR-quiver—the *meshes* found within. In a given AR-quiver there are four possible meshes which can occur throughout the quiver. We list all such in Figure 3.1. These meshes will play an important role when we discuss construction the AR-quiver in the following sections. Moreover, the meshes represent the almost split sequences we discussed earlier. We also remark that the meshes satisfy the following property, for a mesh of the form

$$\begin{tikzcd}
 & M_1 \arrow[dr] \arrow[dd, dash, dashed] & \\
L \arrow[ur] \arrow[dr] \arrow[rr, dashed] & & L' \\
 & M_t \arrow[ur] &
\end{tikzcd}$$

we have that $\underline{\dim} L + \underline{\dim} L' = \sum_{i=1}^{t} \underline{\dim} M_i$.

We now provide some examples, highlighting the AR-quiver with respect to a variety of algebras.

Example 3.3. Let $Q = \; 1 \longrightarrow 2 \longleftarrow 3$, and $A = \mathbb{k}Q$ be the associated path algebra. Then it is easy to compute all indecomposable modules of A;

$$\mathcal{S}(1) = \mathbb{k} \longrightarrow 0 \longleftarrow 0 \quad \mathcal{S}(2) = 0 \longrightarrow \mathbb{k} \longleftarrow 0 \quad \mathcal{S}(3) = 0 \longrightarrow 0 \longleftarrow \mathbb{k}$$

$$\mathcal{P}(1) = \mathbb{k} \longrightarrow \mathbb{k} \longleftarrow 0 \quad \mathcal{I}(2) = \mathbb{k} \longrightarrow \mathbb{k} \longleftarrow \mathbb{k} \quad \mathcal{P}(3) = 0 \longrightarrow \mathbb{k} \longleftarrow \mathbb{k}$$

then the AR-quiver Γ_A (or Γ_Q) is of the form

$$\Gamma_A = \quad \begin{tikzcd}[column sep=small] & \mathcal{P}(3) \arrow[dr] & & \mathcal{S}(1) \\ \mathcal{S}(2) \arrow[ur] \arrow[dr] & & \mathcal{I}(2) \arrow[ur] \arrow[dr] & \\ & \mathcal{P}(1) \arrow[ur] & & \mathcal{S}(3) \end{tikzcd}$$

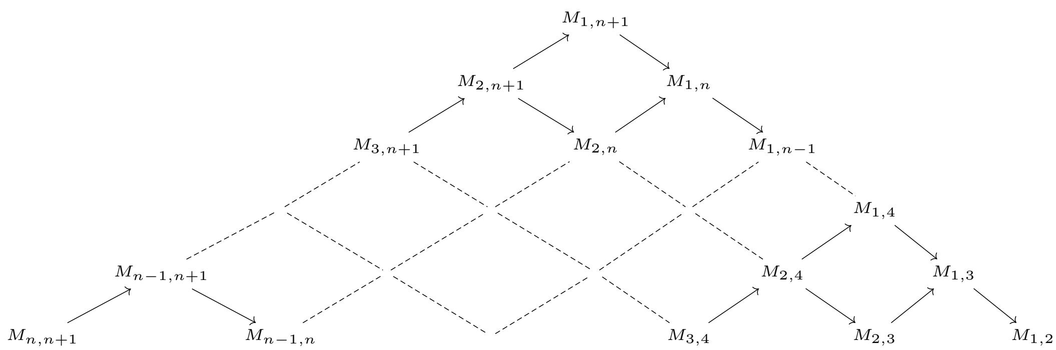
Example 3.4. Clearly we have the $\mathbb{k}$-algebra $M_n(\mathbb{k})$ of $n \times n$ matrices with entries in $\mathbb{k}$, consider the sub-algebra $A \subset M_n(\mathbb{k})$ of upper triangular $n \times n$ matrices. Associated to A is the quiver

$$Q = \vec{\mathbb{A}}_n = \; 1 \xrightarrow{\alpha_1} 2 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_{n-1}} n$$

with left-to-right arrow orientation. It follows that $\mathbb{k}Q = A$ since the set of all paths is of the form:

$$\begin{array}{cccccc} e_1 & e_1\alpha_1 & e_1\alpha_1\alpha_2 & \cdots & e_1\alpha_1 \ldots \alpha_{n-1} \\ & e_1 & e_2\alpha_2 & \cdots & e_2\alpha_2 \ldots \alpha_{n-1} \\ & & \ddots & & \vdots \\ & & & e_{n-1} & e_{n-1}\alpha_{n-1} \\ & & & & e_n \end{array}$$

Let $M_{i,j+1} = \mathcal{S}(i) \oplus \mathcal{S}(i+1) \oplus \cdots \oplus \mathcal{S}(j)$ (e.g., $M_{i,i+1} = \mathcal{S}(i)$), then the AR-quiver of A is shown below.



Note that the bottom row consists of just simple modules, the leftmost diagonal consists of projectives, and the rightmost diagonal consists of injectives.

In the case of $\Delta_Q = \mathbb{D}_n$ we then see instances of the the fourth mesh type.

Example 3.5. Consider the algebra associated to the path algebra of the following quiver.

$$\begin{tikzcd}
& & 3 \\
1 \arrow[r] & 2 \arrow[ur] \arrow[dr] & \\
& & 4
\end{tikzcd}$$

Then, the AR-quiver is shown below.

$$\begin{tikzcd}
\mathcal{S}(3) \arrow[dr] & & (0,1,0,1) \arrow[dr] & & \mathcal{I}(3) \arrow[dr] & & \\
\mathcal{S}(4) \arrow[r] & \mathcal{P}(2) \arrow[ur] \arrow[r] \arrow[dr] & (0,1,1,0) \arrow[r] & (1,2,1,1) \arrow[ur] \arrow[r] \arrow[dr] & \mathcal{I}(4) \arrow[r] & \mathcal{I}(2) \arrow[dr] & \\
& & \mathcal{P}(1) \arrow[ur] & & \mathcal{S}(2) \arrow[ur] & & \mathcal{S}(1)
\end{tikzcd}$$

We can also form AR-quivers for bound quiver algebras.

Example 3.6. Consider the algebra formed by $\Bbbk Q/I$ where

$$Q = \begin{tikzcd} 1 \arrow[r, "\alpha"] & 2 \arrow[r, "\beta"] & 3 \arrow[r, "\gamma"] & 4 \arrow[r, "\delta"] & 5 \end{tikzcd},$$

and $I = \langle \alpha\beta, \gamma\delta \rangle$. Then the AR-quiver is shown below, where $\underline{\dim} M = (0,0,1,1,0)$.

$$\begin{tikzcd}[column sep=small]
 & \mathcal{I}(5) \arrow[dr] & & & & & & \mathcal{I}(2) \arrow[dr] & \\
\mathcal{S}(5) \arrow[ur] & & \mathcal{S}(4) \arrow[dr] & & \mathcal{S}(3) \arrow[dr] & & \mathcal{I}(2) \arrow[ur] & & \mathcal{I}(1) \\
 & & & M \arrow[ur] \arrow[dr] & & \mathcal{I}(3) \arrow[ur] & & & \\
 & & & & \mathcal{I}(4) \arrow[ur] & & & &
\end{tikzcd}$$

The next two sections detail construction of the AR-quiver, we approach this in a variety of ways. The first is the established *Knitting Algorithm* of which is commonly used today, and the second consists of a more algebraic construction derived from a series of lectures given by Gabriel.

3.2 The Knitting Algorithm

We first introduce the algorithm itself.

The Knitting Algorithm. Let Q be a quiver, to construct the AR-quiver Γ_Q we do:

(1) Compute projective indecomposable representations $\mathcal{P}(1), \ldots, \mathcal{P}(n)$.

(2) Draw an arrow $\mathcal{P}(i) \to \mathcal{P}(j) \in \Gamma_Q$ if $j \to i \in Q_1$.

(3) Complete the mesh such that

$$\underline{\dim} L + \underline{\dim} \tau^{-1} L = \sum_{i=1}^{t} \underline{\dim} M_i.$$

(4) Repeat (3) until there are negative values in the dimension vector $\underline{\dim} \tau^{-1} L$.

From Section 2.1.3 we have that computing projective indecomposable representations is rather straightforward, and likewise step (2) of the algorithm is also straightforward. The real difficulty lies in step (3), here the algorithm states to "complete the mesh" which involves computing a new indecomposable representation $\tau^{-1}L$ from L and is described in Figure 3.2. Of course step (4) is also easy enough to follow.

As mentioned the difficulty is in computing $\tau^{-1}L$, given some representation L and the appropriate mesh type. There are three such methods we will discuss; the *Auslander-Reiten translation* τ (and it's inverse τ^{-1}), the Coxeter functor, and lastly we give a geometric interpretation involving regular polygons. The examples presented in this section are all adapted from [Sch14, Chapter 3], but have been expanded on to illustrate the underlying notions at hand.

The Auslander-Reiten translate τ is our main tool, in the AR-quiver it sends the rightmost point of a mesh to the leftmost. We say that the τ-orbit of an indecomposable representation is the set of all representations which can be obtained by applying τ or τ^{-1} repeatedly. In the particular case of $\mathbb{A}_n$ the τ-orbits manifest themselves through the various levels of the AR-quiver, which can been seen in Example 3.3 and Example 3.4.

$$
\begin{tikzcd}
 & M & & & M & \\
L \arrow[ur] & & \arrow[r, dashed] & L \arrow[ur] & & \tau^{-1}L \arrow[from=1-5]
\end{tikzcd}
\tag{3.1}
$$

$$
\begin{tikzcd}
L \arrow[dr] & & \arrow[r, dashed] & L \arrow[dr] & & \tau^{-1}L \\
 & M & & & M \arrow[ur] &
\end{tikzcd}
\tag{3.2}
$$

$$
\begin{tikzcd}
 & M_1 & & & M_1 \arrow[dr] & \\
L \arrow[ur] \arrow[dr] & & \arrow[r, dashed] & L \arrow[ur] \arrow[dr] & & \tau^{-1}L \\
 & M_2 & & & M_2 \arrow[ur] &
\end{tikzcd}
\tag{3.3}
$$

$$
\begin{tikzcd}
 & M_1 & & & M_1 \arrow[dr] & \\
L \arrow[ur] \arrow[r] \arrow[dr] & M_2 & \arrow[r, dashed] & L \arrow[ur] \arrow[r] \arrow[dr] & M_2 \arrow[r] & \tau^{-1}L \\
 & M_3 & & & M_3 \arrow[ur] &
\end{tikzcd}
\tag{3.4}
$$

Figure 3.2: Completing the mesh per the Knitting Algorithm.

3.2.1 Auslander-Reiten Translate τ

Let A be a $\Bbbk$-algebra with $\operatorname{proj} A$ and $\operatorname{inj} A$ denoting the subcategories of $\operatorname{mod} A$ whose objects are projective and injective respectively. We first define the *duality* and then the *Nakayama functor*.

The *duality* is the contravariant functor

$$D = \operatorname{Hom}_{\Bbbk}(-, \Bbbk) : \operatorname{rep} Q \to \operatorname{rep} Q^{\mathrm{op}}.$$

For an object $M \in \operatorname{rep} Q$ the duality sends M to $DM = (DM_i, D\varphi_{\alpha^{\mathrm{op}}})$ where DM_i is the dual vector space of M_i. Hence $DM_i = \operatorname{Hom}_{\Bbbk}(M_i, \Bbbk)$ is the space of linear maps $M_i \to \Bbbk$, and if $\alpha \in Q_1$ then $D\varphi_{\alpha^{\mathrm{op}}}$ is the pullback of φ_α. Thus,

$$D\varphi_{\alpha^{\mathrm{op}}} : DM_{t(\alpha)} \to DM_{s(\alpha)}, \qquad D\varphi_{\alpha^{\mathrm{op}}} : u \mapsto u \circ \varphi_\alpha.$$

For a morphism $f : M \to N$ in $\operatorname{rep} Q$ we have that $Df : DN \to DM$ in $\operatorname{rep} Q^{\mathrm{op}}$, defined by $Df(u) = u \circ f$.

It should be immediate that composition of the duality of Q and the duality of Q^{op} is the identity functor $1_{\operatorname{rep} Q}$ and hence the quasi-inverse of D_Q is $D_{Q^{\mathrm{op}}}$. Furthermore, $D \operatorname{proj} A = \operatorname{inj} A^{\mathrm{op}}$.

Let F be the free representation given in Definition 2.16, it should be clear that the contravariant functor $\operatorname{Hom}(-, F)$ maps $\operatorname{rep} Q$ to $\operatorname{rep} Q^{\mathrm{op}}$. Then, the *Nakayama functor* is defined as the composition of the duality and the contravariant Hom functor, $\nu =$

$D\mathrm{Hom}(-,F) : \mathrm{rep}\, Q \to \mathrm{rep}\, Q$.

$$\begin{tikzcd} \mathrm{rep}\, Q \arrow[r, "\mathrm{Hom}(-,F)"] \arrow[rr, bend right, "\nu"'] & \mathrm{rep}\, Q^{\mathrm{op}} \arrow[r, "D"] & \mathrm{rep}\, Q \end{tikzcd}$$

The Nakayama functor induces an equivalence of categories

$$\nu : \mathrm{proj}\, A \to \mathrm{inj}\, A,$$

where the quasi-inverse is $\nu^{-1} = \mathrm{Hom}_A(DF, -)$ and $DF = \bigoplus_{i \in Q_0} \mathcal{I}(i)$ is the direct sum of indecomposable injective modules. It should also be noted that ν is a *right exact* functor.

Let M be an A-module, the Auslander-Reiten translation τM of M is defined by starting with a minimal projective presentation (a short exact sequence without a leading 0)

$$\mathcal{P}_1 \xrightarrow{p_1} \mathcal{P}_0 \xrightarrow{p_0} M \longrightarrow 0 \ .$$

So $P_0 \xrightarrow{p_0} M$ and $P_1 \xrightarrow{p_0} \ker p_0$ are projective covers and applying the Nakayama functor gives the exact sequence

$$0 \longrightarrow \tau M \longrightarrow \nu\mathcal{P}_1 \xrightarrow{\nu p_1} \nu\mathcal{P}_0 \xrightarrow{\nu p_0} \nu M \longrightarrow 0 \ ,$$

where $\tau M = \ker \nu p_1$ is the *Auslander-Reiten translate* of M. Likewise, beginning with a minimal injective presentation

$$0 \longrightarrow M \xrightarrow{i_0} \mathcal{I}_0 \xrightarrow{i_1} I_1 \ ,$$

and applying ν^{-1}

$$0 \longrightarrow \nu^{-1}M \xrightarrow{\nu^{-1}i_0} \nu^{-1}\mathcal{I}_0 \xrightarrow{\nu^{-1}i_1} \nu^{-1}\mathcal{I}_1 \longrightarrow \tau^{-1}M \longrightarrow 0$$

results in $\tau^{-1}M = \operatorname{coker}\nu^{-1}i_1$. Which is the *inverse Auslander-Reiten translate* of M.

If M is projective then $\tau M = 0$ since $M = \mathcal{P}_0$ in the presentation above and thus $\mathcal{P}_1 = 0$. Similarly, if M is injective then $\tau^{-1}M = 0$.

The following lemma characterizes modules in terms of their Auslander-Reiten translates.

Lemma 3.2. *Let A be a $\Bbbk$-algebra, and M an A-module, then;*

1. $\operatorname{pd}(M) \leq 1$ *if and only if* $\operatorname{Hom}(DA, \tau M) = 0$.

2. $\operatorname{id}(M) \leq 1$ *if and only if* $\operatorname{Hom}(\tau^{-1}M, A) = 0$.

Proof. We prove only the first statement, since the second follows from a dual argument.

Given a minimal projective presentation of M, apply the Nakayama functor. Then, apply the left exact functor ν^{-1} (the quasi-inverse of ν), with the additional isomorphism $\nu^{-1}\nu|_{\operatorname{proj} A} \cong 1_{\operatorname{proj} A}$ results in the commutative diagram, whose rows are exact (here squiggly arrows indicate isomorphism)

$$\begin{tikzcd}
0 \arrow[r] \arrow[d, equal] & \operatorname{Hom}_A(DA, \tau M) \arrow[r] & \nu^{-1}\nu\mathcal{P}_1 \arrow[r, "\nu^{-1}\nu p_1"] \arrow[d, squiggly] & \nu^{-1}\nu\mathcal{P}_0 \arrow[rr] \arrow[d, squiggly] & & 0 \arrow[d, equal] \\
0 \arrow[r] & \ker p_1 \arrow[r] & \mathcal{P}_1 \arrow[r, "p_1"] & \mathcal{P}_0 \arrow[r] & M \arrow[r] & 0
\end{tikzcd}$$

Then the definition of ker implies that we have a morphism f,

$$\begin{tikzcd}
0 \arrow[r] \arrow[d, equal] & \mathrm{Hom}_A(DA, \tau M) \arrow[r] \arrow[d, "f"] & \nu^{-1}\nu\mathcal{P}_1 \arrow[r, "\nu^{-1}\nu p_1"] \arrow[d, rightsquigarrow] & \nu^{-1}\nu\mathcal{P}_0 \arrow[rr] \arrow[d, rightsquigarrow] & & 0 \arrow[d, equal] \\
0 \arrow[r] & \ker p_1 \arrow[r] & \mathcal{P}_1 \arrow[r, "p_1"] & \mathcal{P}_0 \arrow[r] & M \arrow[r] & 0
\end{tikzcd}$$

Then, from Lemma 1.3 (Five Lemma), we have that f is an isomorphism. Therefore, $\mathrm{Hom}_A(DA, \tau M) = 0$ if and only if p_1 is injective if and only if $\mathrm{pd}(M) \leq 1$. □

3.2.2 Coxeter Functor

Let A be a finite-dimensional $\Bbbk$-algebra with finite global dimension with primitive orthogonal idempotents $e_1, \ldots, e_n$ such that $1 = e_1 + \cdots + e_n$. Let the *Cartan matrix* $C_A \in M_n(\mathbb{Z})$ be the $n \times n$ matrix with integer entries c_{ij} given as $c_{ij} = \dim e_j A e_i$.

If A is either a path algebra or a bound quiver algebra then the ith column of C_A is given as $\underline{\dim}\mathcal{P}(i)$ and the ith row of C_A is $\underline{\dim}\mathcal{I}(i)$.

We wish to use the Cartan matrix to define the Coxeter matrix, however to do so we need the inverse of C_A. This leads us to the following theorem, in which it immediately follows that for any algebra with finite global dimension the Cartan matrix is invertible.

Theorem 3.3. *Let A be a bound quiver algebra with finite global dimension, then $\det C_A = 1$ or $\det C_A = -1$.*

Proof. Since A is a bound quiver algebra, we may discuss the simple modules with respect to arbitrary vertices. Let $\mathcal{S}(i)$ be such a simple module, since $\mathrm{gldim}\, A < \infty$

there exists a finite projective resolution

$$0 \longrightarrow \mathcal{P}_{m_i} \longrightarrow \cdots \longrightarrow \mathcal{P}_1 \longrightarrow \mathcal{P}_0 \longrightarrow \mathcal{S}(i) \longrightarrow 0 \ .$$

Then this resolution is exact and hence

$$\underline{\dim}\mathcal{S}(i) = \sum_{j=0}^{m_i} (-1)^j \underline{\dim}\mathcal{P}_j. \tag{1}$$

Since $\mathcal{P}_j$ are direct sums of indecomposable projective representations by Corollary 2.5 and the fact that C_A can be expressed as columns of projective dimension vectors, we know that there is some vector $d_i \in \mathbb{Z}^n$ such that the right hand of (1) is equal to $C_A d_i$. Then, for $\underline{\dim}\mathcal{S}(i)$ we have that the dimension vector is zeros, except at the ith index where it is 1. By varying the vertex i, we can write

$$I_n = C_A D,$$

where D has column vectors $d_1, \ldots, d_n$. Thus C_A is invertible, furthermore since both matrices have integer entries the determinants are two integers whose product is equal to 1. Therefore $\det C_A$ is either 1 or -1. □

Since C_A is invertible, we can construct the following matrix.

Definition 3.7. The *Coxeter matrix* $\phi_A \in M_n(\mathbb{Z})$ is defined to be

$$\phi_A := -C_A^t C_A^{-1},$$

and the *Coxeter transformation* is the linear map from $\mathbb{Z}^n \to \mathbb{Z}^n$ given as $x \mapsto \phi_A(x)$.

Lemma 3.4. *For A a bound quiver algebra the Coxeter matrix induces the equivalence $\phi_A(\underline{\dim}\mathcal{P}(i)) = -\underline{\dim}\mathcal{I}(i)$.*

Proof. We have that $\underline{\dim}\mathcal{P}(i) = C_A(\underline{\dim}\mathcal{S}(i))$ and thus,

$$-\phi_A(\underline{\dim}\mathcal{P}(i)) = -\Big(- C_A^t C_A^{-1}\Big)\Big(C_A(\underline{\dim}\mathcal{S}(i))\Big) = C_A^t(\underline{\dim}\mathcal{S}(i)),$$

but $C_A^t(\underline{\dim}\mathcal{S}(i))$ is just the ith row of C_A which is $\underline{\dim}\mathcal{I}(i)$. □

Example 3.8. Suppose $Q = \; 1 \longrightarrow 2 \longleftarrow 3$, we first compute the projective and injective indecomposable modules, given here as dimension vectors;

$$\mathcal{P}(1) = (1,1,0), \qquad \mathcal{P}(2) = (0,1,0), \qquad \mathcal{P}(3) = (0,1,1),$$

$$\mathcal{I}(1) = (1,0,0), \qquad \mathcal{I}(2) = (1,1,1), \qquad \mathcal{I}(3) = (0,0,1).$$

Then, the Cartan matrix and its inverse are

$$C_A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \qquad C_A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}.$$

Thus the Coxeter matrix is

$$\phi_A = \begin{bmatrix} 0 & -1 & 1 \\ 1 & -1 & 1 \\ 1 & -1 & 0 \end{bmatrix},$$

and for example,

$$\phi_A(\underline{\dim}\mathcal{P}(2)) = \begin{bmatrix} 0 & -1 & 1 \\ 1 & -1 & 1 \\ 1 & -1 & 0 \end{bmatrix} \begin{pmatrix} 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & -1 & -1 \end{pmatrix} = -\underline{\dim}\mathcal{I}(2).$$

We now relate the Coxeter transformation to the Auslander-Reiten translate. Particularly the following result indicates that the Coxeter transformation computes the dimension vector of the Auslander-Reiten translate for a given module over A.

Theorem 3.5. *Let A be a finite-dimensional algebra with finite global dimension. Then, for M an indecomposable A-module;*

1. *If M is non-projective, given the minimal projective presentation*

$$\mathcal{P}_1 \xrightarrow{p_1} \mathcal{P}_0 \xrightarrow{p_0} M \longrightarrow 0 \ ,$$

we have that $\underline{\dim}\tau M = \phi_A\underline{\dim}M - \phi_A\underline{\dim}\ker p_1 + \underline{\dim}\nu M$.

2. *If M is non-injective, given the minimal injective presentation*

$$0 \longrightarrow M \xrightarrow{i_0} \mathcal{I}_0 \xrightarrow{i_1} \mathcal{I}_1 \ ,$$

we have that $\underline{\dim}\tau^{-1}M = \phi_A^{-1}\underline{\dim}M - \phi_A^{-1}\underline{\dim}\operatorname{coker} i_1 + \underline{\dim}\nu^{-1}M$.

Proof. Recall that the Nakayama functor ν maps indecomposable projective representations to indecomposable injective ones, and thus from Lemma 3.4 we see that $\phi_A\underline{\dim}\mathcal{P} = -\underline{\dim}\nu\mathcal{P}$. Similarly, $\phi_A^{-1}\underline{\dim}\mathcal{I} = -\underline{\dim}\nu^{-1}\mathcal{I}$.

We first prove 1., from the minimal projective presentation assumed, we have the exact sequence

$$0 \longrightarrow \ker p_1 \longrightarrow \mathcal{P}_1 \xrightarrow{p_1} \mathcal{P}_0 \xrightarrow{p_0} M \longrightarrow 0\ ,$$

which then yields the relation

$$\underline{\dim} M - \underline{\dim} \ker p_1 = \underline{\dim}\mathcal{P}_0 - \underline{\dim}\mathcal{P}_1.$$

Applying the Coxeter transform gives

$$\phi_A \underline{\dim} M - \phi_A \underline{\dim} \ker p_1 = -\underline{\dim}\nu\mathcal{P}_0 + \underline{\dim}\nu\mathcal{P}_1. \tag{1}$$

The Auslander-Reiten translate of our assumption

$$0 \longrightarrow \tau M \longrightarrow \nu\mathcal{P}_1 \longrightarrow \nu\mathcal{P}_0 \longrightarrow \nu M \longrightarrow 0$$

yields the relation

$$\begin{aligned}\underline{\dim}\nu M - \underline{\dim}\tau M &= -\underline{\dim}\nu\mathcal{P}_0 + \underline{\dim}\nu\mathcal{P}_1\\ \underline{\dim}\tau M &= \underline{\dim}\nu\mathcal{P}_0 - \underline{\dim}\nu\mathcal{P}_1 + \underline{\dim}\nu M\end{aligned}$$

where we apply (1) to get $\underline{\dim}\tau M = \phi_A \underline{\dim} M - \phi_A \underline{\dim} \ker p_1 + \underline{\dim}\nu M$ as desired.

We now prove 2., in a similar manner, from the minimal injective presentation we have the exact sequence

$$0 \longrightarrow M \xrightarrow{i_0} \mathcal{I}_0 \xrightarrow{i_1} \mathcal{I}_1 \longrightarrow \operatorname{coker} i_1 \longrightarrow 0\ ,$$

yielding the relation

$$\underline{\dim}\,\mathrm{coker}\, i_1 - \underline{\dim} M = \underline{\dim}\mathcal{I}_1 - \underline{\dim}\mathcal{I}_0.$$

Applying the (inverse) Coxeter transform gives

$$\phi_A^{-1}\underline{\dim}\,\mathrm{coker}\, i_1 - \phi_A^{-1}\underline{\dim} M = -\underline{\dim}\nu^{-1}\mathcal{I}_1 + \underline{\dim}\nu^{-1}\mathcal{I}_0. \tag{2}$$

The (inverse) Auslander-Reiten translate of our assumption

$$0 \longrightarrow \nu^{-1}M \xrightarrow{\nu^{-1}i_0} \nu^{-1}\mathcal{I}_0 \xrightarrow{\nu^{-1}i_1} \nu^{-1}\mathcal{I}_1 \longrightarrow \tau^{-1}M \longrightarrow 0$$

yields the relation

$$\underline{\dim}\nu^{-1}M - \underline{\dim}\tau^{-1}M = -\underline{\dim}\nu^{-1}\mathcal{I}_1 + \underline{\dim}\nu^{-1}\mathcal{I}_0$$
$$\underline{\dim}\tau^{-1}M = \phi_A^{-1}\underline{\dim} M - \phi_A^{-1}\underline{\dim}\,\mathrm{coker}\, i_1 + \underline{\dim}\nu^{-1}M$$

where we apply (2) to get $\underline{\dim}\tau^{-1}M = \phi_A^{-1}\underline{\dim} M - \phi_A^{-1}\underline{\dim}\,\mathrm{coker}\, i_1 + \underline{\dim}\nu^{-1}M$. $\square$

Corollary 3.6. *Let A be a finite-dimensional algebra with finite global dimension. Then, for M an indecomposable A-module;*

1. *If* $\mathrm{pd}(M) \leq 1$ *and* $\mathrm{Hom}(M, A) = 0$ *then* $\underline{\dim}\tau M = \phi_A \underline{\dim} M$.

2. *If* $\mathrm{id}(M) \leq 1$ *and* $\mathrm{Hom}(DA, M) = 0$ *then* $\underline{\dim}\tau^{-1}M = \phi_A^{-1}\underline{\dim} M$.

This corollary indicates that for non-projective (resp. non-injective) indecomposable modules in a hereditary algebra we can compute the dimension vector of the Auslander-Reiten translate (resp. translate inverse), which can be used to complete the mesh.

The use of the Coxeter matrix/transformation lies in its practicality, when we try to compute the AR-quiver we must find the dimension vectors of all indecomposable representations of a given quiver. Rather than computing minimal projective presentations and applying the Nakayama functor, as the Auslander-Reiten translate tells us to do, it suffices to compute the Coxeter matrix and apply this transformation through a series of matrix multiplications. We now present a couple of examples showcasing the Knitting Algorithm in practice.

Example 3.9. Let Q be the quiver

$$1 \longleftarrow 2 \longleftarrow 3 \longrightarrow 4 \longleftarrow 5\ .$$

Note that $\Delta_Q = \mathbb{A}_5$. We have the indecomposable projective representations;

$$\underline{\dim}\mathcal{P}(1) = (1,0,0,0,0), \qquad \underline{\dim}\mathcal{P}(2) = (1,1,0,0,0), \qquad \underline{\dim}\mathcal{P}(3) = (1,1,1,1,0),$$

$$\underline{\dim}\mathcal{P}(4) = (0,0,0,1,0), \qquad \underline{\dim}\mathcal{P}(5) = (0,0,0,1,1).$$

This completes step (1) of the algorithm, we now need to place the $\mathcal{P}(i)$ in Γ_Q

along with their arrows. This gives us

$$
\begin{tikzcd}[column sep=small]
 & & \mathcal{P}(5) \\
 & \mathcal{P}(4) \arrow[ur] \arrow[dr] & \\
 & & \mathcal{P}(3) \\
 & \mathcal{P}(2) \arrow[ur] & \\
\mathcal{P}(1) \arrow[ur] & &
\end{tikzcd}
$$

In the same manner as Example 3.8, we compute the Coxeter matrix (and its inverse) to be

$$
\phi = \begin{bmatrix} -1 & 1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & -1 & 0 \end{bmatrix} \qquad \phi^{-1} = \begin{bmatrix} 0 & 0 & -1 & 1 & 0 \\ 1 & 0 & -1 & 1 & 0 \\ 0 & 1 & -1 & 1 & 0 \\ 0 & 1 & -1 & 1 & -1 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix}
$$

We now start completing the mesh, beginning with the mesh consisting of $\mathcal{P}(1)$ and $\mathcal{P}(2)$ corresponding to (3.1). This gives $\underline{\dim}\tau^{-1}\mathcal{P}(1) = (0,1,0,0,0)$ with $(1,0,0,0,0)+(0,1,0,0,0) = (1,1,0,0,0) = \underline{\dim}\mathcal{P}(2)$ so the mesh is complete. Likewise, completing the appropriate meshes for $\mathcal{P}(2)$ and $\mathcal{P}(4)$ (meshes (3.1) and (3.3) respectively) gives us $\underline{\dim}\tau^{-1}\mathcal{P}(2) = (0,1,1,1,0)$ and $\underline{\dim}\tau^{-1}\mathcal{P}(4) = (1,1,1,1,1)$, one can easily see that the meshes are complete. These first three iterations give us an updated AR-quiver, we indicate the new meshes as dashed arrows.

$$
\begin{tikzcd}
 & & \mathcal{P}(5) \arrow[dr, dashed] & \\
 & \mathcal{P}(4) \arrow[ur] \arrow[dr] & & (1,1,1,1,1) \\
 & & \mathcal{P}(3) \arrow[ur, dashed] \arrow[dr, dashed] & \\
 & \mathcal{P}(2) \arrow[ur] \arrow[dr, dashed] & & (0,1,1,1,0) \\
\mathcal{P}(1) \arrow[ur] & & (0,1,0,0,0) &
\end{tikzcd}
$$

Continuing on with this process we get the AR-quiver of Q to be

$$
\begin{tikzcd}
 & & \mathcal{P}(5) \arrow[dr] & & \mathcal{I}(1) \arrow[dr] & & \\
 & \mathcal{P}(4) \arrow[ur] \arrow[dr] & & (1,1,1,1,1) \arrow[ur] \arrow[dr] & & \mathcal{I}(2) \arrow[dr] & \\
 & & \mathcal{P}(3) \arrow[ur] \arrow[dr] & & (0,1,1,1,1) \arrow[ur] \arrow[dr] & & \mathcal{I}(3) \\
 & \mathcal{P}(2) \arrow[ur] \arrow[dr] & & (0,1,1,1,0) \arrow[ur] \arrow[dr] & & \mathcal{I}(4) \arrow[ur] \arrow[dr] & \\
\mathcal{P}(1) \arrow[ur] & & \mathcal{S}(2) \arrow[ur] & & (0,0,1,1,0) \arrow[ur] & & \mathcal{I}(5)
\end{tikzcd}
$$

We know where to stop knitting when we reach injective representations, for instance if we tried to compute $\underline{\dim}\tau^{-1}\mathcal{I}(4)$ we would get $(0,0,0,-1,0)$ and there is a negative in the dimension vector, thus $\mathcal{I}(4)$ is the last element in the τ-orbit of $\mathcal{P}(2)$.

We provide another example, where Q is of $\mathbb{D}_n$ type. Recall first the structure of

$\mathbb{D}_n$

$$\begin{tikzcd}[column sep=small]
 & & & & n-1 \\
1 \arrow[r, no head] & 2 \arrow[r, no head] & \cdots \arrow[r, no head] & n-2 \arrow[ur, no head] \arrow[dr, no head] & \\
 & & & & n
\end{tikzcd}$$

In the case of $\mathbb{D}_n$ quivers, the isoclasses of indecomposable representations are given as dimension vectors $\mathbf{d}$ with integer entries of $0, 1, 2$ only. Specifically, for indecomposable representation $\mathbf{d} = (d_1, \ldots, d_n)$ of $\mathbb{D}_n$, if $d_i = 2$ then; i is one of the vertices $2, 3, \ldots, n-2$, i.e. i is not 1 or n, for all vertices j such that $i \leq j \leq n-2$ we have $d_j = 2$, and $d_{i-1} \geq 1$ and $d_{n-1} = d_n = 1$.

Example 3.10. Consider Q as in Example 3.5 where $\Delta_Q = \mathbb{D}_4$

$$\begin{tikzcd}
 & & 3 \\
1 \arrow[r] & 2 \arrow[ur] \arrow[dr] & \\
 & & 4
\end{tikzcd}$$

We have the indecomposable projective representations;

$$\underline{\dim}\mathcal{P}(1) = (1, 1, 1, 1), \qquad \underline{\dim}\mathcal{P}(2) = (0, 1, 1, 1),$$

$$\underline{\dim}\mathcal{P}(3) = (0, 0, 1, 0), \qquad \underline{\dim}\mathcal{P}(4) = (0, 0, 0, 1).$$

Performing step (2) in the Knitting Algorithm results in the initial stretch

$$\begin{tikzcd}
\mathcal{P}(4) \arrow[ddr] & & \\
\mathcal{P}(3) \arrow[dr] & & \\
& \mathcal{P}(2) \arrow[dr] & \\
& & \mathcal{P}(1)
\end{tikzcd}$$

We can then compute the appropriate indecomposable representations using the inverse Coxeter transform, after one iteration of completing meshes we then get

$$\begin{tikzcd}
\mathcal{P}(4) \arrow[ddr] & & (0,1,1,0) \arrow[ddr, dashed] & \\
\mathcal{P}(3) \arrow[dr] & & (0,1,0,1) \arrow[dr, dashed] & \\
& \mathcal{P}(2) \arrow[uur, dashed] \arrow[ur, dashed] \arrow[dr] & & (1,2,1,1) \\
& & \mathcal{P}(1) \arrow[ur, dashed] &
\end{tikzcd}$$

Then the full AR-quiver is

$$\begin{tikzcd}
\mathcal{P}(4) \arrow[ddr] & & (0,1,1,0) \arrow[ddr] & & \mathcal{I}(4) \arrow[ddr] & & \\
\mathcal{P}(3) \arrow[dr] & & (0,1,0,1) \arrow[dr] & & \mathcal{I}(3) \arrow[dr] & & \\
& \mathcal{P}(2) \arrow[uur] \arrow[ur] \arrow[dr] & & (1,2,1,1) \arrow[uur] \arrow[ur] \arrow[dr] & & \mathcal{I}(2) \arrow[dr] & \\
& & \mathcal{P}(1) \arrow[ur] & & \mathcal{S}(2) \arrow[ur] & & \mathcal{I}(1)
\end{tikzcd}$$

Substituting in $\mathcal{S}(i)$ for appropriate projective and injective modules which are also simple, as well as flattening the fourth level to the second results in the presentation shown in Example 3.5.

3.3 $n+3$-Regular Polygons and Type $\mathbb{A}_n$ Quivers

Utilizing regular polygons one can geometrically construct the AR-quiver. In this section we limit ourselves to strictly quivers of $\mathbb{A}_n$ type since the presentation here relies on regular $n+3$-gons. In the case of $\mathbb{D}_n$ type quivers one can also form a geometric construction using arcs of a punctured n-gon [Sch08], however the construction here is a bit more involved.

For a regular polygon a diagonal is a straight line joining two vertices going through the interior, a *triangulation* $\triangle$ of the polygon is a maximal set of diagonals in the polygon which do not intersect.

A full triangulation of our quiver T_Q from Q is constructed as follows; draw a diagonal that cuts off the triangle $\triangle_0$ labelled as 1 (representation vertex 1), if $1 \to 2 \in Q_1$ then draw a diagonal labelled 2 such that 2 is clockwise of diagonal 1 in the triangle $\triangle_1$, if $2 \to 1 \in Q_1$ then draw a diagonal in the same manner, but that is counter-clockwise of 1 in $\triangle_1$. Repeat this for all vertices n.

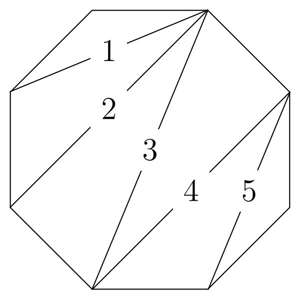
To construct the whole AR-quiver one first computes diagonals representative of the indecomposable projective and injective representations of Q, and then beginning with a projective diagonal we apply the inverse AR-translate (rotate the diagonal counter-clockwise) until we reach the injective diagonal computed ahead of time.

After we have done this for each vertex, i.e. each level since we are working with $\mathbb{A}_n$ quivers, we then fill in the arrows between the various as with the Knitting Algorithm.

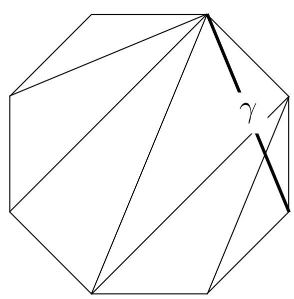
Example 3.11. Consider Q the same as in Example 3.9,

$$1 \longleftarrow 2 \longleftarrow 3 \longrightarrow 4 \longleftarrow 5 \ .$$

We have that $\Delta_Q = \mathbb{A}_5$ and thus we concern ourselves with a regular octagon. Then, the triangulation associated to Q is



And for example the diagonal γ



corresponds to the representation M where $\underline{\dim} M = (0,0,0,1,1)$ since γ crosses lines 4, and 5. Applying the inverse AR-translate then results in the next diagonal

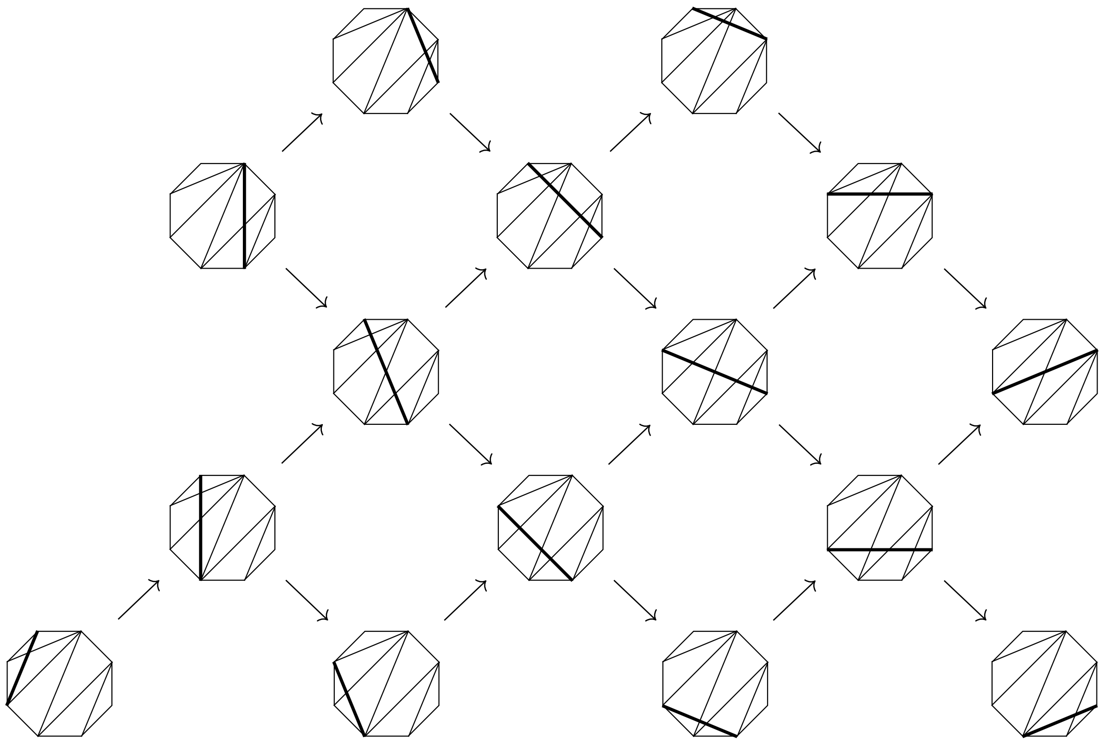
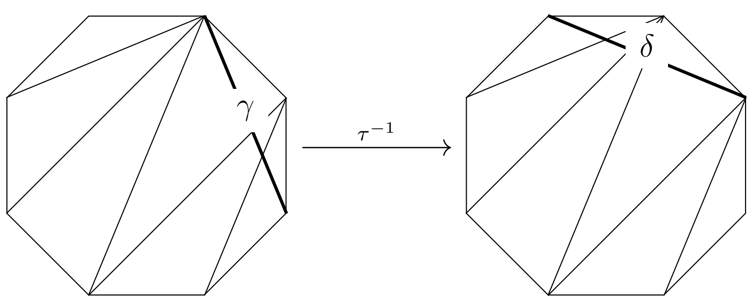


Figure 3.3: Polygonal presentation of AR-quiver for Q in Example 3.9.

$\delta = \tau^{-1}\gamma$, achieved by a counter-clockwise rotation of the diagonal shown below.



Then, the full AR-quiver of Q, in regular $n+3$-gon form is shown in Figure 3.3.

Δ_Q	$\nu : \mathbb{Z} \times Q_0 \to \mathbb{Z} \times Q_0$
$\mathbb{A}_n$	$\nu : (r, i) \mapsto (r + i - 1, n + 1 - i)$
$\mathbb{D}_n$, n even	$\nu : (r, i) \mapsto (r + n - 2, i)$
$\mathbb{D}_n$, n odd	$\nu : (r, i) \mapsto \begin{cases} (r + n - 2, n - 1) & i = n \\ (r + n - 2, n) & i = n - 1 \\ (r + n - 2, i) & i \leq n - 2 \end{cases}$
$\mathbb{E}_6$	$\nu : (r, i) \mapsto \begin{cases} (r + 5, 6) & i = 6 \\ (r + 5, 6 - i) & i \leq 5 \end{cases}$
$\mathbb{E}_7$	$\nu : (r, i) \mapsto (r + 8, i)$
$\mathbb{E}_8$	$\nu : (r, i) \mapsto (r + 14, i)$

Figure 3.4: Nakayama permutation formulas for $\Delta_Q = \mathbb{A}_n, \mathbb{D}_n, \mathbb{E}_{6,7,8}$.

3.4 Infinite Quiver $\mathbb{Z}Q^{\text{op}}$

Here we present Gabriel's construction of the AR-quiver [Gab06], [KS19][1] For a fixed quiver Q, associate to it Q^{op} by reversing all arrows in Q, i.e., for $i \xrightarrow{\alpha} j \in Q_1$ associate $j \xrightarrow{\alpha^*} i \in Q_1^{\text{op}}$. We then construct the infinite quiver $\mathbb{Z}Q^{\text{op}}$ which has $\mathbb{Z} \times Q_0$ (the same as $\mathbb{Z} \times Q_0^{\text{op}}$) as a set of vertices. For an arrow $i \to j \in Q_1$ we associate two arrows in $\mathbb{Z}Q^{\text{op}}$ by $(r, i) \mapsto (r + 1, j)$ and $(r, j) \mapsto (r, i)$. A *slice* of the infinite quiver is a connected full subquiver of $\mathbb{Z}Q^{\text{op}}$ which for each $i \in Q_0$ contains a unique vertex $(r, i) \in \mathbb{Z}Q^{\text{op}}$.

The following result of Gabriel [Gab06, Section 6.5] clearly indicates the relationship the infinite opposite quiver has with the AR-quiver.

Theorem 3.7. *Let Q be a quiver such that Δ_Q is Dynkin. The AR-quiver Γ_Q of Q is identified as a the full subquiver of $\mathbb{Z}Q^{\text{op}}$ formed between the vertices lying between*

[1] I thank Dr. Deniz Kus for his insight into this construction and his assistance in understanding Gabriel's notes.

$S \subset \mathbb{Z}Q^{\mathrm{op}}$ and νS, where ν is the Nakayama permutations listed in Figure 3.4 and S is a subquiver of $\mathbb{Z}Q^{\mathrm{op}}$ isomorphic to Q.

Proof. Fix $x = (r, i) \in \mathbb{Z}Q_0$, there is a slice of $\mathbb{Z}Q$ which admits x as the unique source (resp. sink) called the slice starting (resp. stopping) at x. Let f be a function on the vertices of $\mathbb{Z}Q$, we say f is *additive* if it satisfies

$$f(x) + f(\tau x) = \sum_{\alpha \in Q_1 : t(\alpha) = x} f(s(\alpha)),$$

where the sum is over all arrows whose target is x.

Furthermore, let f_x be the additive function starting at x where $f_x = 1$ on the slice starting at x. In Figure 3.5 we illustrate examples of f_x for Q of $\mathbb{D}_{12}$ type. There are several properties;

1. If S is a slice through x and $y \in S$, if there is a path $x \to y$ in S then $f_x(y) = 1$, otherwise $f_x(y) = 0$.

2. For the appropriate Nakayama permutation we have that f_x is constant with value 1 on the slice which stops at νx.

3. For any slice T through νx, if there is a path from $y \to \nu x$ in T then $f_x(y) = 1$, otherwise $f_x(y) = 0$.

4. For any two vertices $x, y \in \mathbb{Z}Q_0$ we have that $f_{\tau x}(\tau y) = f_x(y)$, $f_{\nu x}(\nu y) = f_x(y)$, and $f_x + f_{\tau^{-1}\nu x} = 0$.

5. The additive functions starting at different vertices of any given slice form a $\mathbb{Z}$-basis of the space of additive functions.

Then, each vertex i of Q_0 determines an integer-valued function g on $\pi_Q \subset \mathbb{N}Q^{\mathrm{op}} \subset \mathbb{Z}Q^{\mathrm{op}}$ where π_Q is the slice of $\mathbb{Z}Q^{\mathrm{op}}$ which is isomorphic to the original presentation

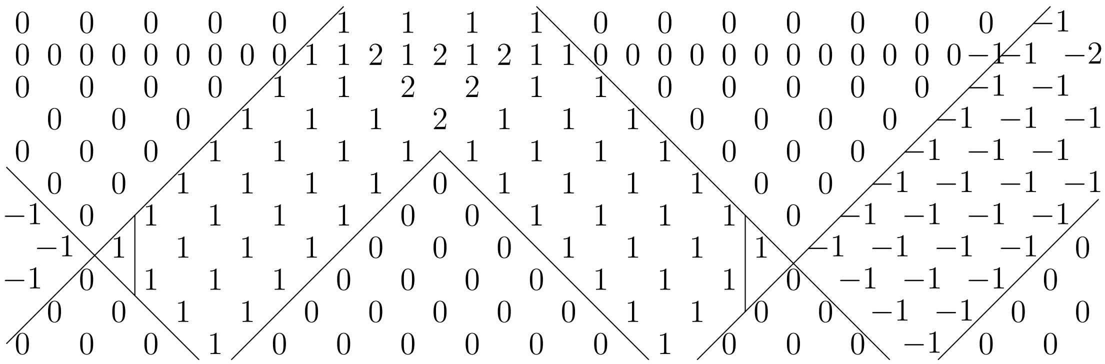


Figure 3.5: Values of the additive functions starting at 1 and stopping at $\nu 1$ for $\mathbb{D}_{12}$ type.

of Q. Like f, this g satisfies

$$g(x) + g(\tau x) = \sum_{\alpha \in Q_1 : t(\alpha) = x} g(s(\alpha)),$$

whenever $x, \tau x \in \mathbb{N}Q^{\mathrm{op}}$. Furthermore, g coincides with f_i on projective vertices, and it is clear that $g = f_i|_{\mathbb{N}Q^{\mathrm{op}}}$. Then, g is positive on $\mathbb{N}Q^{\mathrm{op}}$ but $f_i(\tau^{-1}\nu i) = -f_{\tau^{-1}\nu i}(\tau^{-1}\nu q) = -1$. Therefore, $\tau^{-1}\nu i \notin \mathbb{N}Q^{\mathrm{op}}$ and $\mathbb{N}Q^{\mathrm{op}}$ is between Q^{op} and νQ^{op}. Hence, Q is of finite representation type.

Injective vertices also form a slice of $\mathbb{Z}Q^{\mathrm{op}}$, it can further be seen that for an injective vertex $\mathcal{I}_r$, we have $f_i(\mathcal{I}_r) = 1$ or 0 depending on of there is a path from $\mathcal{I}_r$ to $\mathcal{I}_i$ or not. Hence the injective slice f_i coincides with the additive function stopping at $\mathcal{I}_i$, and $f_i = f_{\nu^{-1}\mathcal{I}_i}$. However, there is only one vertex x between Q^{op} and νQ^{op} such that $f_i = f_{\nu^{-1}x}$, which is of course $x = \nu i$. Therefore we have that $\mathcal{I}_i = \nu i$ and the statement is proved. $\square$

Example 3.12. Consider Q as $1 \longrightarrow 2 \longleftarrow 3$, then Q^{op} is given as $1 \leftarrow 2 \rightarrow 3$. In this case we re-enumerate Q^{op} as $1 \longleftarrow 3 \longrightarrow 2$ so that 1 is a sink in Q and 2 is a sink in $Q_{1^{\mathrm{op}}}$ (the quiver with arrows incoming arrows at vertex 1 reversed).

This gives $\mathbb{Z}Q^{\text{op}}$ to be

$$\begin{tikzcd}
\cdots \arrow[dr] & & (0,2) \arrow[dr] & & (1,2) \arrow[dr] & & \cdots \\
& (0,3) \arrow[ur] \arrow[dr] & & (1,3) \arrow[ur] \arrow[dr] & & (2,3) \arrow[ur] \arrow[dr] & \\
\cdots \arrow[ur] & & (0,1) \arrow[ur] & & (1,1) \arrow[ur] & & \cdots
\end{tikzcd}$$

The unique slice of $\mathbb{Z}Q^{\text{op}}$ containing $(0,3)$ isormorphic to Q^{op} is

$$(0,1) \longleftarrow (0,3) \longrightarrow (0,2)\ .$$

Then, by Theorem 3.7 and the appropriate Nakayama permutation we see that Γ_Q is the subquiver. Hence, the AR-quiver is of the form

$$\begin{tikzcd}
& (0,2) \arrow[dr] & & (1,2) \\
(0,3) \arrow[ur] \arrow[dr] & & (1,3) \arrow[ur] \arrow[dr] & \\
& (0,1) \arrow[ur] & & (1,1)
\end{tikzcd}$$

and computing the appropriate dimension vectors yields

$$\begin{tikzcd}
& (0,1,1) \arrow[dr] & & (1,0,0) \\
(0,1,0) \arrow[ur] \arrow[dr] & & (1,1,1) \arrow[ur] \arrow[dr] & \\
& (1,1,0) \arrow[ur] & & (0,0,1)
\end{tikzcd}$$

which is the AR-quiver of Q.

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Appendix A: Moduli Spaces of Quiver Representations

Motivated by the geometric approach introduced en route to proving Gabriel's theorem in Section 2.3, in this chapter we further define the *moduli space of quiver representations* [Rei08], [Nak16], [Gin09]. The use of quiver moduli is at the forefront of many aspects of quiver theory and broader algebraic geometry research. Some motivating uses include computing quiver Donaldson-Thomas invariants [Arg23] and the representation theory of neural networks (Appendix A.3).

We begin with Appendix A.1 which introduces the high-level idea of a moduli space and a concrete example of lines in $\mathbb{R}^2$. Appendix A.2 then defines the moduli space of quiver representations through applying Geometric Invariant Theory (GIT) to representation spaces of quivers. Lastly, in Appendix A.3 we show one current application of quiver moduli spaces—the representation theory of neural networks.

Most of the material in this chapter requires a more intimate familiarity algebraic geometry. When convenient we detail these new ideas, however some notions which require a stronger background are omitted.

A.1 Moduli Spaces

The general idea of a moduli space is that it allows us to discuss some fixed algebraic/geometric object and define a geometric space which can then be used to classify such objects. If the moduli space has a geometric structure we can then endow the objects with coordinates, we parameterize the objects in such a manner as to establish this coordinate system in the corresponding geometric space. This parameter (or "modulus", hence the name) allows us to determine when two objects in our space are "close".

Consider an arbitrary line in $\mathbb{R}^2$; it consists of two points $a := (x_1, x_2)$ and $b := (x_1', x_2')$. For a given line ℓ let

$$L(\ell) := \sqrt{(x_1' - x_1)^2 + (x_2' - x_2)^2}$$

be the standard formula for the *length* of a line. Naturally, we let $L(\ell)$ be our parameter of the moduli space of all lines in $\mathbb{R}^2$.

It follows then that our moduli space, $\mathcal{M}$, is the positive real number line. Any line has a positive length and as such gets mapped to our space as in Figure A.1.

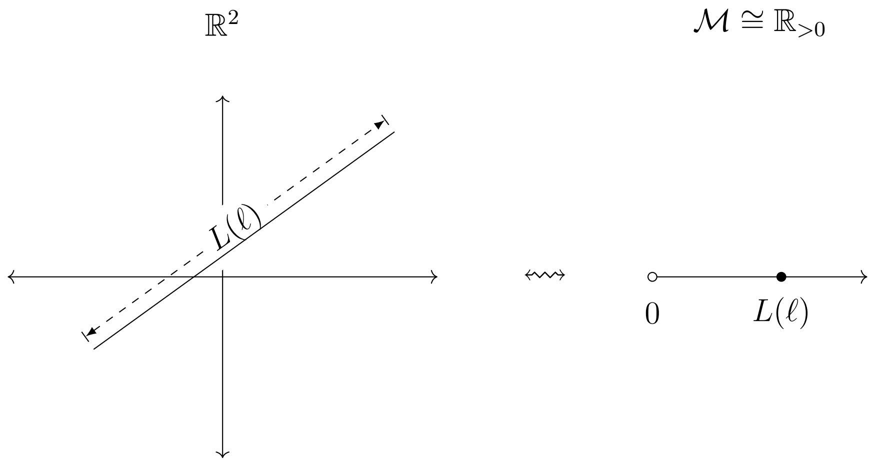

Figure A.1: Line in $\mathbb{R}^2$ mapped to its point in the moduli space $\mathcal{M} \cong \mathbb{R}_{>0}$.

Two lines are considered isomorphic if they have the same length.

The boundary values of our moduli space (the positive real number line) are 0 and ∞, these correspond to "degenerate" lines in $\mathbb{R}^2$. No line can have length 0 or ∞, but we can approach them.

Remark. Limiting ourselves to lines in $\mathbb{R}^2$ which pass through the origin results in the moduli space $\mathcal{M} \cong \mathbb{RP}^1$ which is the projective real line. This construction can be seen either geometrically or topologically. In the geometric sense, our parameter is the positive angle (expressed as radians) $\theta \in [0, \pi)$ of the line with respect to the x_1-axis. In the topological sense, the parameter is a point $s \in S^1 \subset \mathbb{R}^2$ in the unit circle.

More generally, for lines in $\mathbb{R}^{n+1}$ (resp. $\mathbb{C}^{n+1}$) passing through the origin the corresponding moduli space is $\mathbb{RP}^n$ (resp. $\mathbb{CP}^n$).

A.2 Quiver Moduli Spaces

Recall the representation space defined in Section 2.3

$$R_{\mathbf{d}}(Q) := \{M \in \operatorname{rep} Q \mid \underline{\dim} M = \mathbf{d}\} = \bigoplus_{\alpha: i \to j \in Q_1} \operatorname{Hom}\left(\Bbbk^{d_i}, \Bbbk^{d_j}\right),$$

and the change of basis group

$$G_{\mathbf{d}} := \prod_{i \in Q_0} \mathrm{GL}_{d_i}(\Bbbk).$$

We will expand on this representation space by applying GIT to it along with the group $G_{\mathbf{d}}$, allowing us to discuss the moduli space associate to various quiver representations.

A.2.1 Geometric Invariant Theory

We recall some notions of Mumford style GIT in a highly expository manner, further reading can be found in [MFK94] or [Wal17] for a more accessible introduction.

Let V be a vector space and G a reductive algebraic group [Mil15]. Let $\Bbbk[V]^G \subset \Bbbk[V]$ be the subring of invariant polynomials (polynomials $f : V \to \Bbbk$ such that for all $g \in G$ and $v \in V$, $f(gv) = f(v)$). Since G is reductive and the invariant ring is finitely generated, by Hilbert's finiteness theorem we then have that it is the coordinate ring of some variety $V /\!\!/ G := \operatorname{Spec}(\Bbbk[V]^G)$. Embedding $\Bbbk[V]^G$ in $\Bbbk[V]$ results in the morphism $\pi : V \to V /\!\!/ G$, whose fibers contain one closed G-orbit.

Let $V^{\mathrm{st}} \subset V$ be the subset of V consisting of points $v \in V$ such that the orbit Gv is closed and the $\operatorname{Stab}_G(v)$ is zero-dimensional. An early result in GIT asserts that V^{st} is an *open* subset of V and the restriction $\pi|_{V^{\mathrm{st}}} := V^{\mathrm{st}}/G$ gives a morphism whose fibers are the G-orbits in V^{st}.

We now want to extend what we just did to include a character of G; let $\chi : G \to \Bbbk$ be a morphism of algebraic groups. For $f \in \Bbbk[V]$ we say that f is χ-*semi-invariant* if $f(gv) = \chi(g)f(v)$. Let $\Bbbk[V]^{G,\chi}$ be the subring of all χ-semi-invariants and

$$\Bbbk[V]^G_\chi := \bigoplus_{n \geq 0} \Bbbk[V]^{G,\chi^n} \subset \Bbbk[V]$$

be the $\mathbb{Z}$-graded subring of semi-invariants for all powers of χ. It follows that $\Bbbk[V]^G \subset \Bbbk[V]^G_\chi$ since $\Bbbk[V]^G = \Bbbk[V]^{G,\chi^0}$.

An element $v \in V$ is χ-*semi-stable* if there exists some $f \in \Bbbk[V]^{G,\chi^n}$ for some $n \geq 1$ such that $f(v) \neq 0$. Let $V^{\chi-\mathrm{sst}}$ denote the subset of χ-semi-stable points.

If v is a χ-semi-stable point, the orbit Gv is closed in $V^{\chi-\mathrm{sst}}$, and $\operatorname{Stab}_G(v)$ is zero-dimensional then we say v is χ-*stable*. Let $V^{\chi-\mathrm{st}}$ denote the subset of χ-stable points.

We note that both $V^{\chi-\mathrm{sst}}$ and $V^{\chi-\mathrm{st}}$ are open subsets. In Figure A.2 we summarize the relations of the spaces introduced in this section. Where $\operatorname{Proj}(R)$ is the *Proj construction* [Vak23, Chapter 4.5], analogous to Spec. The morphism from $V^{\chi-\mathrm{sst}} /\!\!/ G \twoheadrightarrow V /\!\!/ G$ is projective since the codomain is the degree zero of the domain.

A.2.2 GIT Applied to $R_{\mathbf{d}}(Q)$ and $G_{\mathbf{d}}$

Ultimately to define moduli spaces of quiver representations we need to work with some parameter and the subsequent geometric space. Utilizing the representation space $R_{\mathbf{d}}(Q)$ and the change of base group $G_{\mathbf{d}}$ we can translate the problem of defining a parameter space to the geometric problem of finding a subset U of $R_{\mathbf{d}}(Q)$, an algebraic variety X, and a morphism $\pi : U \to X$ whose fibers are the orbits of $G_{\mathbf{d}}$ in U.

$$
\begin{tikzcd}
V^{\chi-\mathrm{st}} \arrow[r, hook, "\text{open}"] \arrow[d] & V^{\chi-\mathrm{sst}} \arrow[r, hook, "\text{open}"] \arrow[d] & V \arrow[d] \\
V^{\chi-\mathrm{st}}/G \arrow[r, hook, "\text{open}"] \arrow[d, equal] & V^{\chi-\mathrm{sst}} /\!\!/ G \arrow[r, two heads, "\text{projective}"] \arrow[d, equal] & V /\!\!/ G \arrow[d, equal] \\
\mathrm{Proj}(\Bbbk[V]^G_\chi) & \mathrm{Proj}(\Bbbk[V]^G_\chi) & \mathrm{Spec}(\Bbbk[V]^G)
\end{tikzcd}
$$

Figure A.2: GIT space relations.

In the previous section we worked with an arbitrary vector space V and a reductive algebraic group G on V. We want to apply GIT to the vector space $R_{\mathbf{d}}(Q)$ and the group $G_{\mathbf{d}}$, however if we apply GIT directly there will be stabilizers with positive dimension. Note that there is a one-dimensional elements of $G_{\mathbf{d}}$ which act trivially on $R_{\mathbf{d}}(Q)$ by rescaling the bases by the same amount. We consider $\mathbb{P}G_{\mathbf{d}}$ to the factor group of $G_{\mathbf{d}}$ modulo the scalars, i.e., $\mathbb{P}G_{\mathbf{d}} := G_{\mathbf{d}}/\Bbbk^*$ for $\Bbbk^*$ the group of units of $\Bbbk$.

An orbit $\mathcal{O}_M := \mathbb{P}G_{\mathbf{d}}$ is closed in $R_{\mathbf{d}}(Q)$ if and only if $M \in R_{\mathbf{d}}(Q)$ is a *semi-simple representation*, i.e., M is the direct sum of simple representations, by [Art69]. Thus, the quotient variety $R_{\mathbf{d}}(Q) /\!\!/ \mathbb{P}G_{\mathbf{d}}$ parameterizes isomorphism classes of semi-simple representations of Q.

Definition A.1. For a given quiver Q, the *moduli space of semi-simple representations* for a given dimension vector $\mathbf{d} \in \mathbb{Z}^n$ is $\mathcal{M}^{\mathrm{ssimp}}_{\mathbf{d}} := R_{\mathbf{d}}(Q) /\!\!/ \mathbb{P}G_{\mathbf{d}}$.

A representation M of Q is a *Schur representation* if $\mathrm{End}(M) = \Bbbk\,\mathrm{id}_M$. Since $\mathrm{Stab}_{G_{\mathbf{d}}}(M) = \mathrm{Aut}(M)$ we have that $\mathrm{Stab}_{\mathbb{P}G_{\mathbf{d}}}(M)$ is zero-dimensional if and only if M is a Schur representation. Then from GIT we have that the subset $R^{\mathrm{st}}_{\mathbf{d}}(Q)$ is the simple representations of Q with dimension vector $\mathbf{d}$.

Definition A.2. For a given quiver Q, the *moduli space of simple representations* for a given dimension vector $\mathbf{d} \in \mathbb{Z}^n$ is $\mathcal{M}^{\mathrm{simp}}_{\mathbf{d}} := R^{\mathrm{st}}_{\mathbf{d}}(Q)/\mathbb{P}G_{\mathbf{d}}$.

Remark. In the case of quivers without orientated cycles it is clear that $\mathcal{M}^{\mathrm{simp}}_{\mathbf{d}}(Q)$ consists of just the simple indecomposable objects $\mathcal{S}_Q := \{\mathcal{S}(i) \mid i \in Q_0\}$. This follows since for a quiver with no orientated cycles the only simple representations are the indecomposable $\mathcal{S}(i)$ introduced in Section 2.1.3. Hence, to have an "interesting" moduli space, we would need our underlying quiver to have orientated cycles.

As in Appendix A.2.1, we expanded on our initial discussion by introducing characters and referring to χ-(semi-)stability. Since characters of general linear groups are integer powers of the determinant we see that for $m \in \mathbb{Z}^n$ such that $\sum_i^n m_i d_i = 0$, the characters of $\mathbb{P}G_{\mathbf{d}}$ are of the form

$$g = (g_1, \ldots, g_n) \mapsto \prod_{i \in Q_0} \det(g_i)^{m_i}.$$

$$\begin{tikzcd}
R_{\mathbf{d}}^{\Theta-\mathrm{st}}(Q) \arrow[r, hook, "\text{open}"] \arrow[d] & R_{\mathbf{d}}^{\Theta-\mathrm{sst}}(Q) \arrow[r, hook, "\text{open}"] \arrow[d] & R_{\mathbf{d}}(Q) \arrow[d] \\
\mathcal{M}_{\mathbf{d}}^{\Theta-\mathrm{st}}(Q) \arrow[r, hook, "\text{open}"] \arrow[d, equal] & \mathcal{M}_{\mathbf{d}}^{\Theta-\mathrm{sst}}(Q) \arrow[r, two heads, "\text{projective}"] \arrow[d, equal] & \mathcal{M}_{\mathbf{d}}^{\mathrm{ssimp}}(Q) \arrow[d, equal] \\
R_{\mathbf{d}}^{\Theta-\mathrm{st}}(Q)/\mathbb{P}G_{\mathbf{d}} & R_{\mathbf{d}}^{\Theta-\mathrm{sst}}(Q) /\!\!/ \mathbb{P}G_{\mathbf{d}} & R_{\mathbf{d}}(Q) /\!\!/ \mathbb{P}G_{\mathbf{d}}
\end{tikzcd}$$

Figure A.3: Quiver moduli space relations per GIT.

We introduce a linear functional $\Theta : \mathbb{Z}^n \to \mathbb{Z}$, called a *stability*, to define two other moduli spaces of interest. We expand on what it means for representations to be stable after we define such spaces.

We associate a stability to a character χ of $\mathbb{P}G_{\mathbf{d}}$ via

$$\chi_\Theta(g) := \prod_{i \in Q_0} \det(g_i)^{\Theta(\mathbf{d}) - \dim(\mathbf{d} \cdot \Theta_i)}.$$

Let the set of χ_Θ-semi-stable points be denoted as $R_{\mathbf{d}}^{\Theta-\mathrm{sst}}(Q) = R_{\mathbf{d}}^{\chi_\Theta-\mathrm{sst}}(Q)$ and the χ-stable points as $R_{\mathbf{d}}^{\Theta-\mathrm{st}}(Q) = R_{\mathbf{d}}^{\chi_\Theta-\mathrm{st}}(Q)$.

Definition A.3. For a given quiver Q and stability Θ, the *moduli space of stable (resp. semi-stable) representations* for a given dimension vector $\mathbf{d} \in \mathbb{Z}^n$ is $\mathcal{M}_{\mathbf{d}}^{\Theta-\mathrm{st}}(Q) := R_{\mathbf{d}}^{\Theta-\mathrm{st}}(Q)/\mathbb{P}G_{\mathbf{d}}$ (resp. $\mathcal{M}_{\mathbf{d}}^{\Theta-\mathrm{sst}}(Q) := R_{\mathbf{d}}^{\Theta-\mathrm{sst}}(Q) /\!\!/ \mathbb{P}G_{\mathbf{d}}$).

We summarize the GIT applied to the action of $\mathbb{P}G_{\mathbf{d}}$ to the representation space $R_{\mathbf{d}}(Q)$ in Figure A.3.

For $\mathbf{d}$ a non-zero dimension vector of Q define the *slope* to be

$$\mu(d) := \frac{\Theta(\mathbf{d})}{\dim \mathbf{d}}$$

where $\mu(d) \in \mathbb{Q}$. Subsequently, the *slope of a representation* M of Q is given as $\mu(M) := \mu(\underline{\dim} M)$.

Definition A.4. A representation M of Q is called *stable* (resp. semi-stable) if for all non-zero proper subrepresentations $N \subset M$, $\mu(N) < \mu(M)$ (resp. $\mu(N) \leq \mu(M)$). A representation M is called *poly-stable* if it is the direct sum of stable representations with the same slope.

The subcategory $\mathrm{rep}_\mu Q$ of semi-stable representations with slope μ is an abelian subcategory. The indecomposable objects in $\mathrm{rep}_\mu Q$ are in fact the stable objects.

The following result of King [Kin94], connects stability with the moduli spaces we defined in Appendix A.2.2.

Theorem A.1 (King 1994)**.** *For a representation M of Q the following are true;*

1. *M is in $R_{\mathbf{d}}^{\Theta-\mathrm{sst}}(Q)$ if and only if M is semi-stable,*
2. *M is in $R_{\mathbf{d}}^{\Theta-\mathrm{st}}(Q)$ if and only if M is stable,*
3. *$\mathcal{O}_M$ is closed if and only if M is poly-stable.*

Hence, we see that the moduli spaces $\mathcal{M}_{\mathbf{d}}^{\Theta-\mathrm{sst}}(Q)$ parameterize isomorphism classes of poly-stable representations of dimension vector $\mathbf{d}$ and $\mathcal{M}_{\mathbf{d}}^{\Theta-\mathrm{st}}(Q)$ parameterize isomorphism classes of stable representations of dimension vector $\mathbf{d}$.

Further aspects of stability can be investigated in a more algebraic manner, which can be seen in [Rei08, Section 4].

A.3 Representation Theory of Neural Networks

As a "practical" extension of our moduli space discussion we consider the recent works [AJ21] and [ABHR22] which formulates neural networks in terms of quiver representations.

Let Q be a finite acyclic quiver, with $s_1, \ldots, s_p$ the sources of Q and $t_1, \ldots, t_q$ the sinks. Set $\tilde{Q} \subset Q$ to be the subquiver with vertices $\tilde{Q}_0 = Q_0 \setminus \{s_1, \ldots, s_p, t_1, \ldots, t_q\}$. The group

$$G_{\mathbf{d}}(\tilde{Q}) := \{g \in G_{\mathbf{d}}(Q) \mid g_i = 1 \text{ if } i \notin \tilde{Q}_0\}$$

is a subgroup of the change of basis group in the previous section.

Recall that a representation M of Q is *thin* if its dimension vector contains entries of only 0 or 1, we denote the subcategory of thin representations of Q as $\operatorname{thin} Q$.

Definition A.5. Let Q be a connected finite acyclic quiver without multiple arrows. A *neural network* over Q is a pair (W, f) where $W \in \operatorname{thin} Q$ is a thin representation of Q and $f = (f_q)_{q \in \tilde{Q}_0}$ are activation functions $f_q : \Bbbk \to \Bbbk$.

Assume that there are p source vertices, and of the source vertices d are *input vertices* and d' are *bias vertices*, i.e., $p = d + d'$. Let there be q *output vertices* (sinks). An element $\mathbf{x} \in \Bbbk^d$ are called *input vectors* for the network. For a vertex $v \in \tilde{Q}_0$ the *activation output* of the network with respect to an input $\mathbf{x}$ is defined as

$$\mathtt{A}(W, f)_v(\mathbf{x}) = \begin{cases} x_v & \text{if } v \text{ is an input vertex,} \\ 1 & \text{if } v \text{ is a bias vertex,} \\ f_v\left(\sum\limits_{\alpha \in \zeta_v} \left[W_\alpha \cdot \mathtt{A}(W, f)_{s(\alpha)}(\mathbf{x})\right]\right) & \text{otherwise} \end{cases}$$

where $\zeta_v \subset Q_1$ is the set of arrows with target v. We furthermore define the *pre-activation* of a vertex v with respect to the input $\mathbf{x}$ as

$$\mathtt{pre}-\mathtt{A}(W,f)_v(\mathbf{x}) = \begin{cases} 1 & \text{if } v \text{ is an input vertex,} \\ 1 & \text{if } v \text{ is a bias vertex,} \\ \sum\limits_{\alpha\in\zeta_v} \left[W_\alpha \cdot \mathtt{A}(W,f)_{s(\alpha)}(\mathbf{x})\right] & \text{otherwise} \end{cases}$$

Let $t_1, \ldots, t_q$ be the q sink vertices in Q, the *network function* of the neural network is the map

$$\Psi(W,f) : \mathbb{k}^d \to \mathbb{k}^q, \qquad \Psi(W,f) : (\mathbf{x}) \mapsto \big(\mathtt{A}(W,f)_{t_1}(\mathbf{x}), \ldots, \mathtt{A}(W,f)_{t_q}(\mathbf{x})\big)$$

where the right side is clearly in $\mathbb{k}^q$. The *knowledge map* of the network is given as

$$\psi(W,f) : \mathbb{k}^d \to \operatorname{thin} Q, \qquad \psi(W,f) : \mathbf{x} \mapsto W_{\mathbf{x}}^f$$

where $W_{\mathbf{x}}^f = \big(W_{\mathbf{x}}^f\big)_\alpha$ is the collection over all $\alpha \in Q_1$ and

$$\big(W_{\mathbf{x}}^f\big)_\alpha = \begin{cases} W_\alpha \cdot x_{s(\alpha)} & \text{if } s(\alpha) \text{ is an input vertex,} \\ W_\alpha & \text{if } s(\alpha) \text{ is a bias vertex,} \\ W_\alpha \cdot \frac{\mathtt{A}(W,f)_{s(\alpha)}(\mathbf{x})}{\mathtt{pre}-\mathtt{A}(W,f)_{s(\alpha)}(\mathbf{x})} & \text{if } s(\alpha) \text{ is a hidden vertex} \end{cases}$$

The map

$$\hat{\Psi} : \mathcal{M}(Q) \to \mathbb{k}^q, \qquad \hat{\Psi} : [M] \to \Psi(M, \mathrm{id})(1, \ldots, 1)$$

where $\mathcal{M}(Q)$ is the moduli space parameterizing semi-simple representations and (M, id) is the neural network with all identity activation functions.

Armenta and Jodoin [AJ21] showed that the network function is the composition of the knowledge map and $\hat{\Psi}$. That is, the diagram

$$\begin{tikzcd} \mathbb{k}^d \arrow[rr, "\Psi(W,f)"] \arrow[dr, "\psi(W,f)"'] & & \mathbb{k}^q \\ & \mathcal{M}(Q) \arrow[ur, "\hat{\Psi}"'] & \end{tikzcd}$$

commutes. Hence, evaluating the network function of a neural network (W,f) on some fixed input $\mathbf{x}$ is equivalent to evaluating the network function of the neural network $(W_{\mathbf{x}}^f, \mathrm{id})$ on the input $(1, \ldots, 1)$.

Appendix B: Examples in GAP

Here we provide GAP [GAP22] code which works through various examples found within Chapters 2 and 3. Much of what we use here relies on the QPA [QPA22] package, which has been an invaluable asset throughout this project.

B.1 Chapter 2

Example 2.6

```
# Initialize our quiver
gap> Q := Quiver(3, [ [1,2,"a"], [3,2,"b"] ]);;
gap> kQ := PathAlgebra(Rationals, Q);
<Rationals[<quiver with 3 vertices and 2 arrows>]>

# Define representations L and M
gap> LMaps := [  # T:k2->k2, T:(x, y)->(2x, 2y)
>  ["a", [ [2,0],[0,2] ] ],
>  # S:k->k2,  S:(x)->(x, x)
>  ["b", [ [1,1] ] ] ];;
gap> L := RightModuleOverPathAlgebra(kQ, LMaps);
<[ 2, 2, 1 ]>

gap> MMaps := [  # I:k->k, I:(x)|->(x)
>  ["a", [ [1] ] ],
>  # 0:0->k, 0:(0)|->(0)
>  ["b", [ 0, 1] ] ];;
gap> M := RightModuleOverPathAlgebra(kQ, MMaps);
<[ 1, 1, 0 ]>

# Define a morphism f:L->M
gap> MorphismMaps := [ [[10],[-10]], [[5],[-5]], [[0]] ];
[ [ [ 10 ], [ -10 ] ], [ [ 5 ], [ -5 ] ], [ [ 0 ] ] ]
gap> f := RightModuleHomOverAlgebra(L, M, MorphismMaps);
<<[ 2, 2, 1 ]> ---> <[ 1, 1, 0 ]>>

# Construct direct sum of L and M
```

```
gap> TplusS := DirectSumOfQPAModules( [L, M] );
<[ 3, 3, 1 ]>

# Define representation N
gap> NMaps := [  # A:k3->k3, A:(x,y,z)->(2x, 2y, z)
>  ["a", [ [2,0,0],[0,2,0],[0,0,1] ] ],
>  # B:k->k3, B:(x)->(x, x, 0)
>  ["b", [ [1,1,0] ] ] ];;
gap> N := RightModuleOverPathAlgebra(kQ, NMaps);
<[ 3, 3, 1 ]>

# Verify that N is isomorphic to L+M
gap> IsomorphicModules(N, TplusS);
true
```

Example 2.10

```
# Initialize our quiver
gap> Q := Quiver(3, [ [1,2,"a"], [3,2,"b"] ]);;
gap> kQ := PathAlgebra(Rationals, Q);
<Rationals[<quiver with 3 vertices and 2 arrows>]>

# Define representations L and M
gap> LMaps := [  # T:k2->k2, T:(x, y)->(2x, 2y)
>  ["a", [ [2,0],[0,2] ] ],
>  # S:k->k2,  S:(x)->(x, x)
>  ["b", [ [1,1] ] ] ];;
gap> L := RightModuleOverPathAlgebra(kQ, LMaps);
<[ 2, 2, 1 ]>

gap> MMaps := [  # I:k->k, I:(x)|->(x)
>  ["a", [ [1] ] ],
>  # 0:0->k, 0:(0)|->(0)
>  ["b", [ 0, 1] ] ];;
gap> M := RightModuleOverPathAlgebra(kQ, MMaps);
<[ 1, 1, 0 ]>

# Define a morphism g:L->M
gap> MorphismMaps := [ [[10],[-10]], [[5],[-5]], [[0]] ];
[ [ [ 10 ], [ -10 ] ], [ [ 5 ], [ -5 ] ], [ [ 0 ] ] ]
gap> g := RightModuleHomOverAlgebra(L, M, MorphismMaps);
<<[ 2, 2, 1 ]> ---> <[ 1, 1, 0 ]>>

gap> zero := RightModuleOverPathAlgebra(kQ,
```

```
>  [ ["a", [0,0] ],
>  ["b", [0,0] ] ]);
<[ 0, 0, 0 ]>

# Construct kernel and cokernel of g:L->M
gap> kerg := Kernel(g);
<[ 1, 1, 1 ]>
gap> f := KernelInclusion(g);
<<[ 1, 1, 1 ]> ---> <[ 2, 2, 1 ]>>
gap> cokerg := CoKernel(g);
<[ 0, 0, 0 ]>

# Verify sequence 0 -> ker g -> L -> M -> 0 is SE
gap> cat := CatOfRightAlgebraModules(kQ);
<cat: right modules over algebra>
gap> C := FiniteComplex(cat, 1, [g, f]);
0 -> 2:(1,1,1) -> 1:(2,2,1) -> 0:(1,1,0) -> 0
gap> IsExactSequence(C);
true
gap> IsShortExactSequence(C);
true
```

Example 2.15

```
# Initialize our quiver
gap> Q := Quiver(4, [ [1,2,"a"], [1,3,"b"], [1,4,"c"],
>                [2,3,"d"], [3,4,"e"] ]);;
gap> kQ := PathAlgebra(Rationals, Q);
<Rationals[<quiver with 4 vertices and 5 arrows>]>

# Compute simple representations
gap> S := SimpleModules(kQ);;

# Compute projective representations
gap> P := IndecProjectiveModules(kQ);;

# Compute injective representations
gap> I := IndecInjectiveModules(kQ);;

# Print dimension vectors of each indec. rep.
gap> S[1]; S[2]; S[3]; S[4];
<[ 1, 0, 0, 0 ]>
<[ 0, 1, 0, 0 ]>
<[ 0, 0, 1, 0 ]>
```

```
<[ 0, 0, 0, 1 ]>

gap> P[1]; P[2]; P[3]; P[4];
<[ 1, 1, 2, 3 ]>
<[ 0, 1, 1, 1 ]>
<[ 0, 0, 1, 1 ]>
<[ 0, 0, 0, 1 ]>

gap> I[1]; I[2]; I[3]; I[4];
<[ 1, 0, 0, 0 ]>
<[ 1, 1, 0, 0 ]>
<[ 2, 1, 1, 0 ]>
<[ 3, 1, 1, 1 ]>
```

Example 2.18

```
# Initialize our quiver
gap> Q := Quiver(1, [ [1,1,"a"] ]);;
gap> kQ := PathAlgebra(Rationals, Q);
<Rationals[<quiver with 1 vertices and 1 arrows>]>

# Observe dimension of kQ
gap> Dimension(kQ);
infinity
```

Example 2.19

```
# Initialize our quiver
gap> Q := Quiver(2, [ [1,2,"a"], [1,2,"b"] ]);;
gap> kQ := PathAlgebra(Rationals, Q);
<Rationals[<quiver with 2 vertices and 2 arrows>]>

# Define representation M
gap> MMaps := [ # T:k2->k3, T:(x,y)->(x,y,0)
>          ["a", [ [1,0,0],[0,1,0] ] ],
>          # S:k2->k2, S:(x,y)->(0,x,y)
>          ["b", [ [0,1,0],[0,0,1] ] ] ];;
gap> M := RightModuleOverPathAlgebra(kQ, MMaps);
<[ 2, 3 ]>


# Define morphism maps (identity)
gap> f1 := [ [1,0],[0,1] ];;
```

```
gap> f2 := [ [1,0,0],[0,1,0],[0,0,1] ];;

gap> f := RightModuleHomOverAlgebra(M, M, [f1, f2]);
<<[ 2, 3 ]> ---> <[ 2, 3 ]>>

# Compute EndM and verify its dimension
gap> Dimension(EndOverAlgebra(M));
1
```

Example 2.22

```
# Initialize our quiver
gap> Q := Quiver(2, [ [1,1,"a"], [1,2,"b"] ]);;
gap> kQ := PathAlgebra(Rationals, Q);
<Rationals[<quiver with 2 vertices and 2 arrows>]>

# Assign generator variables and make relations
gap> AssignGeneratorVariables(kQ);
#I  Assigned the global variables [ v1, v2, a, b ]

gap> rho1 := a^2*b;;
gap> rho2 := a^4;;

# Create ideal
gap> I := Ideal(kQ, [rho1, rho2]);
<two-sided ideal in <Rationals[<quiver with 2 vertices and 2 arrows>]>,
  (2 generators)>

# Is I admissible
gap> IsAdmissibleIdeal(I);
true
```

Example 2.26

```
# Initialize our quiver
gap> Q := Quiver(6, [ [1,2,"a"], [2,4,"b"], [1,3,"c"],
>                  [3,4,"d"], [4,5,"e"], [5,6,"f"] ]);;
gap> kQ := PathAlgebra(Rationals, Q);
<Rationals[<quiver with 6 vertices and 6 arrows>]>

# Assign generator variables and make relations
gap> AssignGeneratorVariables(kQ);
#I  Assigned the global variables [ v1,v2,v3,v4,v5,v6,a,b,c,d,e,f ]
```

```
gap> rho1 := a*b - c*d;;
gap> rho2 := b*e;;
gap> rho3 := d*e*f;;

gap> R := [rho1, rho2, rho3];
[ (1)*a*b+(-1)*c*d, (1)*b*e, (1)*d*e*f ]

# Create ideal
gap> gb := GBNPGroebnerBasis(R, kQ);;
gap> I := Ideal(kQ, gb);;
gap> GroebnerBasis(I, gb);
<complete two-sided Groebner basis containing 3 elements>

# Create bound alg.
gap> A := kQ / I;
<Rationals[<quiver with 6 vertices and 6 arrows>]/
<two-sided ideal in <Rationals[<quiver with 6 vertices and 6 arrows>]>,
  (3 generators)>>

# Compute projective representations
gap> P := IndecProjectiveModules(A);;

# Print dimension vectors of each indec. proj. rep.
gap> P[1]; P[2]; P[3]; P[4]; P[5]; P[6];
<[ 1, 1, 1, 1, 0, 0 ]>
<[ 0, 1, 0, 1, 0, 0 ]>
<[ 0, 0, 1, 1, 1, 0 ]>
<[ 0, 0, 0, 1, 1, 1 ]>
<[ 0, 0, 0, 0, 1, 1 ]>
<[ 0, 0, 0, 0, 0, 1 ]>
```

Example 2.28

```
# Initialize our quiver
gap> Q := Quiver(3, [ [1,2,"a"], [3,2,"b"] ]);;
gap> kQ := PathAlgebra(Rationals, Q);
<Rationals[<quiver with 3 vertices and 2 arrows>]>

# Get unit form of kQ
gap> B := TitsUnitFormOfAlgebra(kQ);;
gap> q := SymmetricMatrixOfUnitForm(B);;
gap> Display(q);
[ [   2,  -1,   0 ],
```

```
  [  -1,   2,  -1 ],
  [   0,  -1,   2 ] ]
```

B.2 Chapter 3

For the examples in Chapter 3, we omit some of the computations done in GAP as they are quite repetitive and often tedious. A series of helper functions has been written and included in the appendix as Appendix B.3.

Example 3.3

```
# Read helper functions
gap> Read("lib/arq.g");;

# Initialize our quiver
gap> Q := Quiver(3, [ [1,2,"a"], [3,2,"b"] ]);;
gap> kQ := PathAlgebra(Rationals, Q);
<Rationals[<quiver with 3 vertices and 2 arrows>]>

# Call AR-quiver method
gap> cache := ARQuiver(Q);;
gap> dims := cache[2];;

# Print dims.
gap> DisplayAR(dims);
[ 1, 1, 0 ] [ 0, 0, 1 ]
[ 0, 1, 0 ] [ 1, 1, 1 ]
[ 0, 1, 1 ] [ 1, 0, 0 ]
```

Example 3.8

```
# Initialize our quiver
gap> Q := Quiver(3, [ [1,2,"a"], [3,2,"b"] ]);;
gap> kQ := PathAlgebra(Rationals, Q);
<Rationals[<quiver with 3 vertices and 2 arrows>]>

# Compute indec proj. and inj. reps.
gap> P := IndecProjectiveModules(kQ);;
gap> I := IndecInjectiveModules(kQ);;
gap> P[1]; P[2]; P[3];
<[ 1, 1, 0 ]>
<[ 0, 1, 0 ]>
```

```
<[ 0, 1, 1 ]>
gap> I[1]; I[2]; I[3];
<[ 1, 0, 0 ]>
<[ 1, 1, 1 ]>
<[ 0, 0, 1 ]>

# Print Cartan matrix (and inverse)
gap> C := TransposedMat(CartanMatrix(kQ));;
gap> CInverse := Inverse(C);;
gap> Display(C); Display(CInverse);
[ [  1,  0,  0 ],
  [  1,  1,  1 ],
  [  0,  0,  1 ] ]
[ [   1,   0,   0 ],
  [  -1,   1,  -1 ],
  [   0,   0,   1 ] ]

# Print Coxeter matrix
gap> Coxeter := TransposedMat(CoxeterMatrix(kQ));;
gap> Display(Coxeter);
[ [   0,  -1,   1 ],
  [   1,  -1,   1 ],
  [   1,  -1,   0 ] ]

# Verify assertation about P(2) and I(2)
gap> Coxeter * DimensionVector(P[2]) = -1 * DimensionVector(I[2]);
true
```

Example 3.9

```
# Initialize our quiver
gap> Q := Quiver(5, [ [2,1,"a"], [3,2,"b"], [3,4,"c"], [5,4,"d"] ]);;
gap> kQ := PathAlgebra(Rationals, Q);
<Rationals[<quiver with 5 vertices and 4 arrows>]>

# Compute indec. proj. reps.
gap> P := IndecProjectiveModules(kQ);;
gap> P[1]; P[2]; P[3]; P[4]; P[5];
<[ 1, 0, 0, 0, 0 ]>
<[ 1, 1, 0, 0, 0 ]>
<[ 1, 1, 1, 1, 0 ]>
<[ 0, 0, 0, 1, 0 ]>
<[ 0, 0, 0, 1, 1 ]>
```

```
# Compute Coxeter matrix
gap> Coxeter := TransposedMat(CoxeterMatrix(kQ));;
gap> CoxeterInverse := Inverse(Coxeter);
[ [ 0, 0, -1, 1, 0 ], [ 1, 0, -1, 1, 0 ], [ 0, 1, -1, 1, 0 ],
  [ 0, 1, -1, 1, -1 ], [ 0, 0, 0, 1, -1 ] ]

# Compute first three iterations of knitting alg.
gap> CoxeterInverse * DimensionVector(P[1]);
[ 0, 1, 0, 0, 0 ]
gap> CoxeterInverse * DimensionVector(P[2]);
[ 0, 1, 1, 1, 0 ]
gap> CoxeterInverse * DimensionVector(P[4]);
[ 1, 1, 1, 1, 1 ]
```

Example 3.10

```
# Initialize our quiver
gap> Q := Quiver(4, [ [1,2,"a"], [2,3,"b"], [2,4,"c"] ]);;
gap> kQ := PathAlgebra(Rationals, Q);
<Rationals[<quiver with 4 vertices and 3 arrows>]>

# Compute indec. proj. reps.
gap> P := IndecProjectiveModules(kQ);;
gap> P[1]; P[2]; P[3]; P[4];
<[ 1, 1, 1, 1 ]>
<[ 0, 1, 1, 1 ]>
<[ 0, 0, 1, 0 ]>
<[ 0, 0, 0, 1 ]>

# Iterate once with Coxeter transform
gap> CoxeterInverse := Inverse(TransposedMat(CoxeterMatrix(kQ)));;
gap> CoxeterInverse * DimensionVector(P[4]);
[ 0, 1, 1, 0 ]
gap> CoxeterInverse * DimensionVector(P[3]);
[ 0, 1, 0, 1 ]
gap> CoxeterInverse * DimensionVector(P[2]);
[ 1, 2, 1, 1 ]
```

B.3 Files

AR-Quiver.g

```
# Series of helper functions which compute the AR-quiver
# Authored by Neelam Venkata Prasad Akula, 2023
```

```
# Compute next rep. in a level
tauL := function(L, mat, kQ)
local dimVec, d;
  if IsInjectiveModule(L) then return fail; fi;

      dimVec := DimensionVector(L)*mat;
      for d in dimVec do
         if d < 0 then return fail; fi;
      od;

      return RightModuleOverPathAlgebra(kQ, dimVec, []);
end;

# Compute whole level of proj. rep.
computeLevel := function(p, mat, kQ)
local L, level, tL;
      L := p;
      level := [];
      Add(level, L);

      while not tauL(L, mat, kQ) = fail do
         tL := tauL(L, mat, kQ);
         Add(level, tL);
         L := tL;
      od;

      return level;
end;

# Compute all dim. vecs. of indec. reps.
ARQuiverDimensions := function(dims)
local dimensions, level, l, levels;

      dimensions := [];

      for level in dims do
         levels := [];
         for l in level do
          Add(levels, DimensionVector(l));
         od;
    Add(dimensions, levels);
      od;

      return dimensions;
```

```
end;

# Compute AR-quiver
ARQuiver := function(Q)
local kQ, P, mat, dimensions, p, level, adj, dims, cache;

      kQ := PathAlgebra(Rationals, Q);;
      P := IndecProjectiveModules(kQ);;

      mat := Inverse(CoxeterMatrix(kQ));;

      dimensions := [];

      for p in P do
         level := computeLevel(p, mat, kQ);
    Add(dimensions, level);
      od;

      adj := AdjacencyMatrixOfQuiver(Q);
      dims := ARQuiverDimensions(dimensions);
      cache := [adj, dims];

      return cache;
end;

# Print AR-quiver from dims.
DisplayAR := function(dims)
local level, dim;
      for level in dims do
         for dim in level do
      Print(dim, " ");
         od;
         Print("\n");
      od;
end;
```

Nakayama-Perm.g

```
# Function to compute Nakayama Permutation of an element (r,i) of ZQ
# Authored by Neelam Venkata Prasad Akula, 2023

nakayamaPerm := function(DynkinType, n, r, i)
local R, I;

  if DynkinType = "A" then
      # (r,i) -> (r+i-1, n+1-i)
      R := r + i - 1;
```

```
        I := n + 1 - i;
        return [R, I];

    elif DynkinType = "D" then
        if n mod 2 = 1 then
            if i = n then
                # (r,i) -> (r+n-2, n-1)
                R := r + n - 2;
                I := n - 1;

            elif i = n-1 then
                # (r,i) -> (r+n-2,n)
                R := r + n - 2;
                I := n;
            fi;
        else
            # (r,i) -> (r+n-2, i)
            R := r + n - 2;
            I := i;
        fi;

        return [R, I];

    elif DynkinType = "E" then
        if n = 6 then
            if i = 6 then
                # (r,i) -> (r+5,6)
                R := r + 5;
                I := 6;
                return [R, I];

            else
                # (r,i) -> (r+5,6-i)
                R := r + 5;
                I := 6 - i;
                return [R, I];
            fi;
        elif n = 7 then
            # (r,i) -> (r+8,i)
            R := r + 8;
            I := i;
            return [R, I];

        elif n = 8 then
            # (r,i) -> (r+14,i)
```

```
          R := r + 14;
          I := i;
          return [R, I];
      else return fail; fi;
  else return fail; fi;
end;
```