MATH7313 STUDENT LECTURE 3

NEELAM AKULA AND JARED SMITH

Material adapted from §4.3.4 – 4.4.3 [1].

1. Traffic Jam

In the previous lecture, we introduced the topic of traffic dynamics. For traffic density $\rho(x,t)$ and initial density $g(x)$, we can use conservation laws to derive the following equations:

$$\begin{aligned} \rho_t + q(\rho)_x &= 0 \\ \rho(x,0) &= g(x) \end{aligned} \tag{1}$$

with flux function $q(\rho) = \rho v_m (1 - \rho/\rho_m)$. We solved the above equation by means of characteristic curves. That is, given a point (x,t) and a curve $x(t) = q'(g(x_0))t + x_0$ satisfying $\rho(x(t),t) = g(x_0)$, we can find a solution of the form

$$\rho(x,t) = g(q'(g(x_0))t - x). \tag{2}$$

We applied this method the "green light problem", with initial density

$$g(x) = \begin{cases} \rho_m & x \le 0 \\ 0 & x > 0 \end{cases}. \tag{3}$$

The discontinuity of g initially prevented us from using characteristics to find a solution, but we were able to resolve this problem by approximating g with a family of continuous functions g_ε which converge to g as $\varepsilon \to 0$.

In this lecture, we will be looking at a related problem, which will require us to once again modify our characteristic curve approach in order to find a solution.

We will once again be solving the equation given in (1), but this time we will be using the initial density

$$g(x) = \begin{cases} 1/8\rho_m & x < 0 \\ \rho_m & x > 0 \end{cases}. \tag{4}$$

We can imagine that this initial density describes a traffic jam with bumper-to-bumper traffic starting at $x = 0$.

Calculating $q'(\rho) = v_m(1 - 2\rho/\rho_m)$, we find that the characteristic curves for this problem are given by

$$x = \begin{cases} 3/4 v_m t + x_0 & x_0 < 0 \\ -v_m t + x_0 & x_0 > 0 \end{cases}. \tag{5}$$

The main observation here is that, since x has a negative slope for $x > 0$ and positive slope for $x < 0$, the characteristic curves will intersect each other. This creates a problem, since following a point (x,t) back to its origin in the manner we have been will make ρ a multi-valued function,

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which doesn't make sense. To resolve this issue, we will have to modify our conservation law in order to allow for discontinuous ρ. Recall that our conservation law is given in integral form to be

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{x_1}^{x_2}\rho(x,t)dx = -q(\rho(x_2,t)) + q(\rho(x_1,t)) \tag{6}$$

Now, assume that ρ is smooth except along a smooth curve $x = s(t)$. Then we can reformulate (6) as

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\{\int_{x_1}^{s(t)}\rho(y,t)dy + \int_{s(t)}^{x_2}\rho(y,t)dy\right\} = -q(\rho(x_2,t)) + q(\rho(x_1,t)) \tag{7}$$

We can evaluate the left-hand side using the fundamental theorem of calculus. Let $F = \int \rho(x,t)dy$. Then $\frac{\mathrm{d}F}{\mathrm{d}t} = \frac{\partial F}{\partial t} + \frac{\partial F}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t}$. We calculate

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{x_1}^{s(t)}\rho(y,t)dy = \frac{\partial F}{\partial t}\bigg|_{(x_1,t)}^{(s(t),t)} - \frac{\mathrm{d}y}{\mathrm{d}t}\frac{\partial F}{\partial y}\bigg|_{(x_1,t)}^{(s(t),t)} \tag{8}$$

$$= \int_{x_1}^{s(t)}\rho_t(y,t)dy + \dot{s}\rho_-(s(t),t), \tag{9}$$

where $\rho_-(s(t),t) \equiv \lim_{y\to s(t)^-}\rho(y,t)$. A similar calculation gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{s(t)}^{x_2}\rho(y,t)dy = \int_{s(t)}^{x_2}\rho_t(y,t)dy - \dot{s}\rho_+(s(t),t), \tag{10}$$

with $\rho_+(s(t),t)$ defined analogously to $\rho_-(s(t),t)$. Since x_1, x_2 are arbitrary, let $x_1 \to s(t)$ from the right and $x_2 \to s(t)$ from the left so that the integral terms in the previous two expressions disappear. Plugging in these new expressions into (6), our new conservation law can be written

$$\dot{s}(\rho_-(s(t),t) - \rho_+(s(t),t)) = -q(\rho_+(s(t),t)) + q(\rho_-(s(t),t)). \tag{11}$$

From this conservation law, we can put conditions on $\dot{s}$,

$$\dot{s} = \frac{q(\rho_-(s(t),t) - q(\rho_+(s(t),t))}{\rho_-(s(t),t) - \rho_+(s(t),t)} \tag{12}$$

This condition on $\dot{s}$ is called the **Rankine-Hugoniot jump condition**, and the solution ρ satisfying these conditions is called a **shock wave**. Indeed, recall that s is a smooth curve along which ρ is discontinuous, and that ρ is smooth except at these points. So then s is a line of discontinuity, and we can define ρ to be the piecewise function,

$$\rho(x,t) = \begin{cases}\rho_-(x,t) & x < s(t)\\ \rho_+(x,t) & x > s(t)\end{cases}. \tag{13}$$

Then we need only calculate ρ,ρ_+, s.

$$\rho_- = g(x_0) = 1/8\rho_m \tag{14}$$

$$\rho_+ = \rho_m \tag{15}$$

$$q(\rho_-) = \frac{\rho_m}{8}v_m\left(1 - \frac{1/8\rho_m}{\rho_m}\right) = 7/64\rho_m v_m \tag{16}$$

$$q(\rho_+) = 0 \tag{17}$$

$$\dot{s} = \frac{7/64\rho_m v_m}{-7/8\rho_m} = \frac{-1}{8}v_m \tag{18}$$

Since $\dot{s}$ is constant and $s(0) = 0$ (following from $\rho(x,0) = g(x)$), s is a linear equation given by

$$s = \frac{-1}{8} v_m t. \tag{19}$$

Finally, we can write the solution to the traffic jam problem posed at the beginning of this section:

$$\rho(x,t) = \begin{cases} 1/8\rho_m & x < \frac{-1}{8} v_m t \\ \rho_m & x > \frac{-1}{8} v_m t \end{cases} \tag{20}$$

Note that, since s is decreasing, the shock propagates backward along x as t increases. This is what we should expect; the front car will slow down as it approaches the traffic jam at $x = 0$, causing the cars behind him slow down in turn. Fortunately, despite encountering some difficulties applying the method of characteristics, the solution we have arrived at seems to make physical sense.

2. Characteristics Revisited

As we have seen in the previous two examples, the method of characteristics will sometimes fail for certain initial densities. In this section, we discuss the criteria for which the method of characteristics will yield a solution. For the differential equation

$$\begin{aligned} u_t + q(u)_x &= 0 \\ u(x,0) &= g(x) \end{aligned} \tag{21}$$

the method of characteristics gives the solution u to be the wave equation

$$u(x,t) = g(x - q'(g(\xi))t) \tag{22}$$

Along a characteristic curve, we have $u = g(\xi)$, so we can define u implicitly as

$$G(x,t,u) = u - g(x - q'(u)t) \tag{23}$$

The implicit function theorem states that $G(x,t,u)$ implicitly defines a function in u under the condition that

$$G_u \neq 0 \tag{24}$$

$$\implies -1 \neq t g'(x - q'(u)t) q''(u) \tag{25}$$

Noting that $g'(x - q'(u)t) = g(\xi), q''(u) = q''(g(\xi))$, Equation (25) gives us an important sufficient condition for the existence of solutions: $g'(\xi)q''(g(\xi))$ must be positive. Or, in other words, the two function in the product must have the same sign. This is a very reasonable result. Indeed, since the slope of characteristic curves is given by $q'(g(\xi))$, we calculate

$$\frac{\mathrm{d}}{\mathrm{d}\xi} q'(g(\xi)) = g'(\xi) q''(g(\xi)). \tag{26}$$

Thus, $g'(\xi)q''(g(\xi))$ positive implies that the slope of the characteristic curves is increasing in x, which will guarantee that the curves are non-intersecting.

On the other hand, if $g'(\xi)q''(g(\xi))$ is negative, we may still have a unique solution in some region. However, we would expect a shock wave due to the intersecting characteristics. Therefore, in order to determine the region on which the implicit function theorem guarantees the existence of a solution u, we need to find the *breaking time* t_s corresponding to the appearance of the first shock. Referring back to (25), we have the t_s will be the smallest t for which this condition fails. Or in other words,

$$t_s = \min_{\xi \in [x_1, x_2]} \frac{1}{-g'(\xi) q''(g(\xi))} \tag{27}$$

We notice that $z(\xi) = -g'(\xi)q''(g(\xi))$ is a strictly positive function, since the two functions have opposite signs. Therefore, if z attains a maximum value only at some $\xi_M \in [x_1, x_2]$, then

$$t_s = \frac{-1}{g'(\xi_M)q''(g(\xi_M))}. \tag{28}$$

The corresponding x_s lies on a characteristic curve, so we can also calculate

$$x_s = \frac{-q'(g(\xi_M))}{g'(\xi_M)q''(g(\xi_M))} + \xi_M. \tag{29}$$

In conclusion, the existence of a solution u is guaranteed for $t \in [0, t_s)$. Now let's apply this analysis to an example:

$$\begin{aligned} &u_t + (1-2u)u_x = 0 \qquad (30)\\ &u(x,0) = 1/2 \arctan(\pi x) \end{aligned}$$

We have $q = u - u^2, q' = 1 - 2u, q'' = -2$. The characteristic curves, as usual, are given by $x = q'(g(\xi))t + \xi$. The solution given by the method of characteristics is

$$G(x,t,u) = u - 1/2 \arctan(\pi(x - (2u - u)t)). \tag{31}$$

Since g is an increasing function and q is concave, our solution has a shock. First, note that $z = -g'(\xi)q''(g(\xi)) = \frac{\pi}{\pi^2\xi^2+1}$ is maximized by $\xi_M = 0$. We can then calculate

$$t_s = \frac{1}{-g'(\xi_M)q''(g(\xi_M))} \tag{32}$$

$$= \frac{1}{\pi}. \tag{33}$$

So the solution implicitly defined in (31) is unique and exists on $0 \le t < 1/\pi$. We can extend this region to $t > 1/\pi$, but will require us to introduce a shock, similarly to the previous section. This notion will be made exact in the next section.

3. Weak Solutions and the Rankine-Hugoniot Condition

From the previous sections we have one natural question:

Q1. *In what sense is the differential equation satisfied across a shock (or more generally a separation curve) where the constructed solution is* not *differentiable?*

which leads to:

Q2. *Is the solution unique?*

Our goal in this section is to answer **Q1** and **Q2**. To do so we need a more flexible solution, where the derivative of the solution does not appear. Recall the problem.

Problem 1.
$$\begin{cases} u_t + q(u)_x = 0, & x \in \mathbb{R}, t > 0 \\ u(x,0) = g(x), & x \in \mathbb{R} \end{cases}$$
where q is smooth and g bounded.

If $u \in C^1(\mathbb{R} \times [0,\infty))$ is smooth and a solution to Problem 1 we say that u is a *classical solution*.

Now let v be a smooth function with compact support in $\mathbb{R} \times [0,\infty)$, it vanishes outside any compact set contained in our domain $\mathbb{R} \times [0,\infty)$, and is called a *test function*. Multiplying the differential equation by v and integrating on $\mathbb{R} \times (0,\infty)$ yields the equation

$$\int_0^\infty \int_{\mathbb{R}} \left[u_t + q(u)_x\right] v dx dt = \int_0^\infty \int_{\mathbb{R}} u_t v dx dt + \int_0^\infty \int_{\mathbb{R}} q(u)_x v dx dt = 0. \tag{34}$$

Integrating by parts with respect to t on the first part yields

$$\begin{aligned} \int_0^\infty \int_{\mathbb{R}} u_t v dx dt &= -\int_0^\infty \int_{\mathbb{R}} u v_t dx dt - \int_{\mathbb{R}} u(x,0) v(x,0) dx \\ &= -\int_0^\infty \int_{\mathbb{R}} u v_t dx dt - \int_{\mathbb{R}} g(x) v(x,0) dx \end{aligned}$$

and with respect to x on the second part yields

$$\int_0^\infty \int_{\mathbb{R}} q(u)_x v dx dt = -\int_0^\infty \int_{\mathbb{R}} q(u) v_x dx dt$$

so eq. (34) becomes

$$\int_0^\infty \int_{\mathbb{R}} [u v_t + q(u) v_x] dx dt + \int_{\mathbb{R}} g(x) v(x,0) dx = 0. \tag{35}$$

Which holds for any test function v.

Proposition 3.1. *A function $u \in C^1(\mathbb{R} \times [0,\infty))$ is a classical solution of Problem 1 if and only if eq. (35) holds for all test functions v.*

Proof. We have shown the forwards direction in the derivation of eq. (35).

Now suppose that u is a smooth function that satisfies eq. (35) for all test functions v. Then integrating by parts (first with respect to x then by t) gives us

$$\int_0^\infty \int_{\mathbb{R}} \left[u_t + q(u)_x\right] v dx dt + \int_{\mathbb{R}} \left[g(x) - u(x,0)\right] v(x,0) dx = 0$$

for all v. Consider now two cases on our test function v: vanishing and not vanishing at $t = 0$. If v vanishes then

$$\int_{\mathbb{R}} \left[g(x) - u(x,0)\right] v(x,0) dx = 0 \implies u_t q(u)_x = 0$$

if v does not vanish then

$$\int_{\mathbb{R}} \left[g(x) - u(x,0)\right] v(x,0) dx = 0.$$

Therefore $u(x,0) = g(x)$ and u is a classical solution. □

Relaxing our constraints on u still works with the results of eq. (35) as there is no derivative of u in it. Hence it suffices to consider functions u which are only bounded.

Definition 3.1 (Weak Solution)**.** Let u be a bounded function in $\mathbb{R} \times [0, \infty)$. Then u is a *weak solution* to Problem 1 if eq. (35) holds for every test function v in $\mathbb{R} \times [0, \infty)$.

Given this definition we still do not understand the behaviour a weak solution has across a discontinuity curve.

Definition 3.2. A function $u : \mathbb{R} \times [0, \infty) \to \mathbb{R}$ is a *piecewise-C^1* function if there exists a finite number of C^1 curves Γ_j, $1 \leq j \leq N$, of equation $x = s_j(t)$ defined in some interval $I_j \subseteq [0, \infty)$ such that:

(1) u is C^1 outside of $\bigcup_j \Gamma_j$;
(2) and u is C^1 up to each Γ_j from both sides.

That is, for each Γ_j, either u is continuous or it undergoes a jump discontinuity, and the jumps $u_+(s_j(t), t) - u_-(s_j(t), t)$ are continuous on I_j.

In the class of piecewise-C^1 functions, the only admissible discontinuities in $t > 0$ are those which satisfy the previously introduced Rankine-Hugoniot condition.

Theorem 3.2. *Let $u : \mathbb{R} \times [0, \infty) \to \mathbb{R}$ be a piecewise-C^1 function, then u is a weak solution of Problem 1 if and only if:*

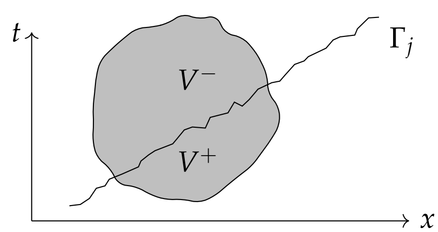
(1) u is a classic solution in the region(s) where u is C^1;
(2) and the Rankine-Hugoniot jump condition

$$\dot{s}_j(t) = \frac{q\left(u_+(s_j(t), t)\right) - q\left(u_-(s_j(t), t\right)}{u_+(s_j(t), t) - u_-(s_j(t), t)}$$

holds along each discontinuity line $\Gamma_j \cap (\mathbb{R} \times (0, \infty))$.

For this theorem, we examine weak solutions by using test functions supported in $\mathbb{R} \times (0, \infty)$.

Proof. ($\Rightarrow$) Suppose that u is a weak solution of Problem 1, we know that u is a classical solution in any open set where u is C^1. To see (2) let Γ_j be one discontinuity curve (with equation $x = s_j(t)$), V be an open set contained in the half-plane of $t > 0$, and partition V into two disjoint open domains, V^- and V^+, by Γ_j:



In both V^+ and V^-, u is a classical solution. Let v be a test function supported in the compact set $K \subset V$, where $K \cap \Gamma_j$ is nonempty. Since $v(x, 0) = 0$ we have

$$0 = \int_0^\infty \int_{\mathbb{R}} [uv_t + q(u)v_x]\, dxdt = \int_{V^+} [uv_t + q(u)v_x]\, dxdt + \int_{V^-} [uv_t + q(u)v_x]\, dxdt.$$

Integrating by parts, noting that $v = 0$ on $\partial V^+ \setminus \Gamma_j$, yields

$$\begin{aligned}\int_{V^+} [uv_t + q(u)v_x]\, dxdt &= -\int_{V^+} [u_t + q(u)_x]\, vdxdt + \int_{\Gamma_j} [u_+ n_2 + q(u_+)n_1]\, vdl \\ &= \int_{\Gamma_j} [u_+ n_2 + q(u_+)n_1]\, vdl\end{aligned}$$

where u_+ is the value of u on Γ_j from the V^+ side, $n = (n_1, n_2)$ is the outward unital normal vector on ∂V^+, and dl is the arc length on Γ_j. Similarly, for u_- the value of u on Γ_j from the V^- side we have

$$\int_{V^-} [uv_t + q(u)v_x]\, dxdt = -\int_{\Gamma_j} [u_- n_2 + q(u_-)n_1]\, vdl.$$

Therefore we have

$$\int_{\Gamma_j} \Big([q(u_+) - q(u_-)]\, n_1 + [u_+ - u_-]\, n_2 \Big) vdl = 0.$$

Since u is piecewise-C^1 the jumps $[q(u_+) - q(u_-)]$ and $[u_+ - u_-]$ are both continuous on Γ_j, and since v is an arbitrary test function we have that

$$[q(u_+) - q(u_-)]\, n_1 + [u_+ - u_-]\, n_2 = 0 \tag{36}$$

on Γ_j. If u is continuous across Γ_j then eq. (36) is satisfied, otherwise if $u_+ \neq u_-$ we have that

$$n = (n_1, n_2) = \frac{1}{\sqrt{1 + (\dot{s}_j(t))^2}}(-1, \dot{s}_j(t))$$

and hence expanding eq. (36) gives us that

$$\dot{s}_j(t) = \frac{q\left[u_+\left(s_j(t), t\right)\right] - q\left[u_-\left(s_j(t), t\right)\right]}{u_+\left(s_j(t), t\right) - u_-\left(s_j(t), t\right)}$$

which is the Rankine-Hugoniot condition, along the smooth curve Γ_j.

($\Leftarrow$) The converse direction follows immediately, as shown in the previous section. □

Hence, by Theorem 3.2, the solutions of the traffic jam (and the green light) problem are exactly the weak solutions (since shock waves satisfy the Rankine-Hugoniot condition). Furthermore, using the definition of piecewise-C^1 functions we have a suitable answer to **Q1**. What about **Q2**?

Example 3.3. Consider a flux of particles along the x-axis, all moving at constant speed. Let $u(x,t)$ be the velocity field, i.e. the speed of a particle located at x at time t. If $x = x(t)$ is the path of a particle then its velocity at t is given by $\dot{x}(t) = u(x(t), t)$, which is a constant. Therefore,

$$0 = \frac{d}{dt}u(x(t),t) = u_t(x(t),t) + u_x(x(t),t)\dot{x}(t) = u_t(x(t),t) + u_x(x(t),t)u(x(t),t)$$

hence u satisfies *Burgers' equation*:

$$u_t + uu_x = u_t + \left(\frac{u^2}{2}\right)_x = 0$$

which is a conservation law with $q(u) := u^2/2$. Note that $q' = u$ and $q'' = 1$, hence q is strictly convex. Defining initial conditions $u(x,0) = g(x)$ where

$$g(x) := \begin{cases} 0, & x < 0 \\ 1, & x > 0 \end{cases} \qquad \text{yields the system} \qquad \begin{cases} u_t + q(u)_x = 0, & x \in \mathbb{R}, t > 0 \\ u(x,0) = g(x), & x \in \mathbb{R} \end{cases}$$

Then the characteristics are the straight lines $x = g(x_0)t + x_0$. Therefore, $u = 0$ if $x < 0$ and $u = 1$ if $x > t$. The region $S := \{0 < x < t\}$ is not covered by characteristic. Similar to the green light problem, we connect the states 0 and 1 with a rarefaction wave. Since $q'(u) = u$ we have $r(s) = (q')^{-1}(s) = s$ and we construct the weak solution:

$$u(x,t) = \begin{cases} 0, & x \leq 0 \\ x/t, & 0 < x < t \\ 1, & x \geq t. \end{cases}$$

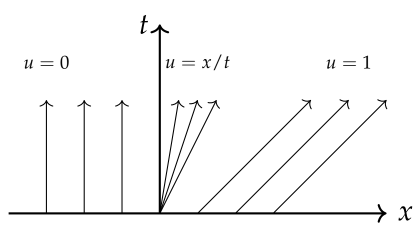
However, u is *not the unique weak solution*—there exists a further shock wave solution! Since

$$u_- = 0, u_+ = 1, q(u_-) = 0, q(u_+) = \frac{1}{2} \implies \dot{s}(t) = \frac{q(u_+) - q(u_-)}{u_+ - u_-} = \frac{1}{2}$$

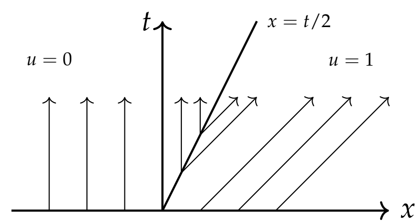
by the Rankine-Hugoniot condition. Given the discontinuity at $x = 0$, the shock curve starts at $s(0) = 0$ and is the straight line $x = t/2$, hence we have another weak solution:

$$w(x,t) = \begin{cases} 0, & x < t/2 \\ 1, & x > t/2. \end{cases}$$

The following pictures illustrate the two solutions.



Rarefaction Wave



Nonphysical Shock

Since we now know we cannot guarantee uniqueness there is a new question which arises:

Q3. *If uniqueness is not guaranteed, how do you select the "correct" solution?*

where by "correct" we mean the solution which makes the most physical sense, which will be answered in a future lecture.

Suggested Exam Question

Problem. *A family of curves $\phi(x,t,\xi) = 0$ is said to* admit an envelope *if there exists a curve $\psi(x,t) = 0$ which is tangent at each of its points to a curve in the family. Let u be a solution to the differential equation*

$$\begin{cases} u_t + q(u)_x = 0 \\ u(x,0) = g(x), \end{cases}$$

show that the family of characteristics

$$x(u,t) = q'(u)t + \xi, \qquad \xi \in [a,b]$$

admits an envelope if $q''(g(\xi))$ and $g'(\xi)$ have opposite signs.

Additionally, show that the point with minimum time in the envelope is (t_s, x_s), where t_s is the breaking time defined above.

Hint: If an envelope exists for a family of curves $\phi(x,t,\xi)$, then its parametric equations are given by solving the following system for x and t:

$$\begin{cases} \phi(x,t,\xi) = 0, \\ \phi_\xi(x,t,\xi) = 0. \end{cases}$$

Solution. Solving the system given in the problem statement gives the parametric equations for $\psi(x,t)$ to be

$$x = \frac{q'(g(\xi))}{-q'(g(\xi))g'(\xi)} + \xi, \qquad t = \frac{1}{-q''(g(\xi))g'(\xi)}.$$

Then, for each ξ, $\psi(x,t)$ is tangent to the curve $\phi(x,t,\xi) = x - q'(g(\xi))t - \xi$ at the points given above. To find the point of minimum time on the envelope, we calculate the minimum of the parametric equation for t given above, which is exactly t_s. Plugging the corresponding ξ_M minimizing t into x once again gives us exactly x_s.

References

[1] S. Salsa, *Partial differential equations in action: From modelling to theory*, UNITEXT, Springer International Publishing, 2016.