Math 448A - Quiver Representations

(Notes by A.V.P. Neelam)

1.1 Defns. & Examples

Defn. A Quiver $Q = (Q_0, Q_1, s, t)$ is a quadruple.

- Q_0 - a set of vertices
- Q_1 - a set of arrows
- $s, t : Q_1 \longrightarrow Q_0$ set mappings source & target resp.

Convention: Use $\mathbb{N}$ to index over the set of vertices & use greek letters for the arrows b/n them.

Let k be an algebraically closed arbitrary field.

Defn. A representation $M = (M_i, \varphi_\alpha)_{i \in Q_0, \alpha \in Q_1}$ is a collection of k-vector spaces & a collection of k-linear maps. Where the M_i are corresponding to each vertex, and the linear maps are of the form:

$$\varphi_\alpha : M_{s(\alpha)} \longrightarrow M_{t(\alpha)}$$

Defn. Let M, M' be rep. of Q. A morphism of reps $f : M \longrightarrow M'$ is a collection of linear maps $f_i : M_i \longrightarrow M'_i$ s.t. $\forall \alpha \in Q_1$ the following diagram commutes.

$$\begin{array}{ccc} M_i & \xrightarrow{\varphi_\alpha} & M_j \\ {\scriptstyle f_i}\downarrow & & \downarrow{\scriptstyle f_j} \\ M'_i & \xrightarrow{\varphi'_\alpha} & M'_j \end{array}$$

That is, $f_j \circ \varphi_\alpha (m) = \varphi'_\alpha \circ f_i (m) \ \forall m \in M_i$.

1.2 Direct Sums & Indecomposable Reps.

<u>Defn.</u> Let $M=(M_i,\varphi_\alpha)$, $M'=(M_i',\varphi_\alpha')$ be reps. of Q.

Then,

$$M \oplus M' = \left(M_i \oplus M_i', \begin{bmatrix} \varphi_\alpha & 0 \\ 0 & \varphi_\alpha' \end{bmatrix}\right)$$

is a rep. of Q. Called the <u>Direct Sum</u> of M & M'.

<u>Example</u>: Let $Q=(\{1,2,3\},\{\alpha,\beta\},s,t)$ s.t. Q is of the form

$$Q:\quad 1 \longrightarrow 2 \longleftarrow 3$$

Consider the reps:

$$M:\quad k \xrightarrow{1} k \xleftarrow{0} 0$$

$$M':\quad k^2 \xrightarrow[\begin{bmatrix}1&1\\0&1\end{bmatrix}]{} k^2 \xleftarrow[\begin{bmatrix}1\\1\end{bmatrix}]{} k$$

Then,

$$M \oplus M':\quad k^3 \xrightarrow[\begin{bmatrix}1&0&0\\0&1&1\\0&0&1\end{bmatrix}]{} k^3 \xleftarrow[\begin{bmatrix}0&0\\0&1\\0&1\end{bmatrix}]{} k$$

<u>Defn.</u> A representation M of Q is called <u>Indecomposable</u> if There are no non-zero reps N, L s.t. $M = N \oplus L$.

Thm (Krull-Schmidt). Let Q be a quiver & let $M \in \operatorname{rep} Q$.

Then
$$M \simeq M_1 \oplus \cdots \oplus M_t$$
where the $M_i \in \operatorname{rep} Q$ are indecomposable & unique up to order.

Pf. M indecom. trivial. Sps. $M = M' \oplus M''$, use induction on M' & M''. Indecomposability follows easily. Uniqueness is in other texts. $\square$

Defn. A category $\mathcal{C}$ consists of objects, morphisms, & binary operation of composistion of morphisms.

- $Ob(\mathcal{C})$ the class of objects $X, Y, Z \in Ob(\mathcal{C})$.
- $Hom_{\mathcal{C}}$ the class of morphisms $f \in Hom_{\mathcal{C}}(X,Y)$ for $X, Y \in Ob(\mathcal{C})$
  $f : X \to Y$, $Hom(X,Y)$ is the class of all morphisms from X to Y.

Let $X, Y, Z \in Ob(\mathcal{C})$. Then $\left(Hom(X,Y) \times Hom(Y,Z)\right) \to Hom(X,Z)$ is s.t. $(f,g) \mapsto g \circ f$. Satisfying the following.

(i) If $f : W \to X$, $g : X \to Y$, $h : Y \to Z$ then $h \circ (g \circ f) = (h \circ g) \circ f$.

(ii) If $X \in Ob(\mathcal{C})$, $\exists\ 1_X \in Hom(X,X)$ s.t. $\forall\ f \in Hom(X,Y)$ and $g \in Hom(Z,X)$ we have
$$f \circ 1_X = f \quad \text{and} \quad 1_X \circ g = g.$$

1.3 Kernels, Coker, & Exact Sequences.

Lin. Alg. Review

Sps f is a linear map b/n vector spaces $f: V \longrightarrow V'$

We have the following notions of kernels & cokernels.

$\ker f := \{ v \in V \mid f(v) = 0 \}$ subspace of V

$\operatorname{coker} f := V'/\operatorname{im} f = \{ v' + f(V) \mid v' \in V' \}$ quotient space of V'

We translate these ideas to the notion of Quivers.

Kernels

Fix Q w/ reps $M = (M_i, \varphi_\alpha)$ & $M' = (M'_i, \varphi'_\alpha)$ and

Let $f: M \longrightarrow M'$ be a morphism of reps.

For each $i \in Q_0$ let $L_i = \ker f_i$ & For each $\alpha \in Q_1$

let $\psi_\alpha : L_i \longrightarrow L_j$ s.t. $\psi_\alpha = \varphi_\alpha \restriction L_i$ $\left(\psi_\alpha(x) = \varphi_\alpha(x), \forall x \in L_i \right)$

Show ψ_α is well-defined:

NTS $x \in L_i \Rightarrow \psi_\alpha(x) \in L_j \equiv \varphi_\alpha(x) \in \ker f_j$.

Since $f \in \operatorname{Hom}(M, M')$ we have $f_j \varphi_\alpha(x) = \varphi'_\alpha f_i(x) = 0$

Since $x \in \ker f_i$. $\therefore$ ψ_α is well-defined. $\square$

Defn The rep. $\ker f = (L_i, \psi_\alpha)$ is called the <u>kernel</u> of f.

The inclusions $\mathrm{incl}_i : \ker f_i \hookrightarrow M_i$, form an injective morphism of reps. $(\mathrm{incl}_i)_{i \in Q_0} : \ker f \hookrightarrow M_i$.

Cokernels

Fix Q w/ reps $M = (M_i, \varphi_\alpha)$ & $M' = (M'_i, \varphi'_\alpha)$ and

Let $f: M \to M'$ be a morphism of reps.

For each $i \in Q_0$ let $N_i = \operatorname{coker} f_i = M'_i / f_i(M_i)$ &

For each $\alpha \in Q_1$ let $\chi_\alpha : N_i \to N_j$ s.t.

$$\chi_\alpha(m' + f_i(M_i)) = \varphi'_\alpha(m'_i) + f_j(M_j)$$

Show χ_α is well-defined:

Sps $m'_i, m''_i \in M'_i$ s.t. $m'_i + f_i(M_i) = m''_i + f_i(M_i)$

$\Rightarrow m'_i - m''_i \in f(M_i)$

$\Rightarrow \varphi'_\alpha(m'_i) - \varphi'_\alpha(m''_i) = \varphi_\alpha(m'_i - m''_i) \in \varphi'_\alpha f_i(M_i)$

But $\varphi'_\alpha f_i(M_i) = f_j(M_i) \subset f_j(M_j)$. It follows that

$$\chi_\alpha(m'_i + f_i(M_i)) = \chi_\alpha(m''_i + f_i(M_i)) \qquad \square$$

__Defn.__ The rep. $\operatorname{coker} f = (N_i, \chi_\alpha)$ is the __cokernel__ of f.

The projections $\mathrm{proj}_i : M'_i \twoheadrightarrow \operatorname{coker} f_i$ induce a surjective morphism of reps. $(\mathrm{proj}_i)_{i \in Q_0} : M' \twoheadrightarrow \operatorname{coker} f$.

Category Theory Interpretation/Generalization (optional)

We can generalize the results of kernels & cokernels from linear algebra $\to$ Quivers $\to$ Categories.

1)* Let $M \xrightarrow{g} N$ be a morphism. A kernel of g is a morphism $L \xrightarrow{f} M$ s.t. $gf=0$. And given any $v \in \mathrm{Hom}(X,M)$ s.t. $gv=0$ there $\exists!\, u \in \mathrm{Hom}(X,L)$ s.t. $fu=v$. We say v factors through f:

$$\begin{tikzcd}
 & X \arrow[dl, dashed, "\exists! u"'] \arrow[d, "v"] & \\
L \arrow[r, "f"] & M \arrow[r, "g"] & N
\end{tikzcd}$$

2)* Let $L \xrightarrow{f} M$ be a morphism. A cokernel of f is a morphism $M \xrightarrow{g} N$ s.t. $gf=0$. And given any $v \in \mathrm{Hom}(M,X)$ s.t. $vf=0$ there $\exists!\, u \in \mathrm{Hom}(N,X)$ s.t. $ug=v$. We say v factors through g:

$$\begin{tikzcd}
L \arrow[r, "f"] & M \arrow[r, "g"] \arrow[dr, "v"'] & N \arrow[d, dashed, "\exists! u"] \\
 & & X
\end{tikzcd}$$

Defn. An abelian k-category is such that:

1) $\mathrm{Hom}(M,N)$ is a k-vector space, comp. of morphisms is bilinear.
2) $\mathcal{C}$ has direct sums, & a zero object s.t. $1_0 \in \mathrm{Hom}(0,0)$ is the zero of the vector space $\mathrm{Hom}(0,0)$.
3) Each $f \in \mathrm{Hom}(M,N)$ has a kernel $i: K \to M$ and a cokernel $p: N \to C$ s.t. $\mathrm{coker}\, i \simeq \ker p$. (1st iso Thm).

$\mathrm{rep}\, Q$ is an abelian k-category.

* we can verify our defs of ker & coker for Quivers using our defs of 1) & 2). This is in the text.

<u>Defn.</u> A rep. L is called a <u>subrepresentation</u> of a rep. M if there is an injective morphism $i: L \hookrightarrow M$. Here the <u>quotient representation</u> M/L is defined to be $\operatorname{coker} i$.

<u>Thm (1st iso thm).</u> If $f: M \to N$ is a morphism of reps, then

$$\operatorname{im} f \simeq M/\ker f.$$

<u>Pf.</u> Let $M = (M_i, \varphi_\alpha)$, then $\operatorname{im} f = (f(M_i), \psi_\alpha)$ w/ maps $\psi_\alpha(f_i(m_i)) = f_j \varphi_\alpha(m_i)$ for all arrows.

On the other hand — $M/\ker f = (M_i/\ker f_i, \chi_\alpha)$ where $\chi_\alpha(m_i + \ker f_i) = \varphi_\alpha(m_i) + \ker f_j$.

Since each f_i is a lin. map we have an isomorphism of vector spaces.

$$\bar{f}_i : \left(M_i/\ker f_i\right) \to f_i(M_i) \text{ s.t. } \bar{f}_i : (m_i + \ker f_i) \mapsto f_i(m_i).$$

Moreover — for each $\alpha \in Q$, $i \xrightarrow{\alpha} j$ we have $\psi_\alpha \bar{f}_i = \bar{f}_j \varphi_\alpha$

thus, $\bar{f}$ is a morphism of reps between $\operatorname{im} f$ & $M/\ker f$. $\square$

Exact Sequences

__Defn.__ A sequence of morphisms $L \xrightarrow{f} M \xrightarrow{g} N$ is called __exact at M__ if $\operatorname{im} f = \ker g$. A sequence is called __exact__ if it is exact at all M_i.

__Defn.__ A __short exact sequence__ is an exact sequence of the form

$$0 \longrightarrow L \xrightarrow{f} M \xrightarrow{g} N \longrightarrow 0$$

Short exact iff f injective, $\operatorname{im} f = \ker g$, g surjective.

__Defn.__ A morphism $f : L \to M$ is called a __section__ if there exists a morphism $h : M \to L$ s.t. $h \circ f = 1_L$.
A morphism $g : M \to N$ is called a __retraction__ if there exists a morphism $h : N \to M$ s.t. $g \circ h = 1_N$.

We say that a short exact seq. __splits__ if f is a section.

__Example.__ Sps $Q = 1 \longrightarrow 2$ w/ reps

$$\underset{S(2)}{0 \longrightarrow k} \qquad \underset{M}{k \xrightarrow{1} k} \qquad \underset{S(1)}{k \longrightarrow 0}$$

(1) $\quad 0 \longrightarrow S(2) \xrightarrow{f} M \xrightarrow{g} S(1) \longrightarrow 0$

w/ $f = (f_1, f_2) = (0, 1)$ & $g = (g_1, g_2) = (1, 0)$

is a short exact seq. __Not split__ since $\nexists\, h \neq 0 \in \operatorname{Hom}(M, S(2))$

Example (cont.)

(2) $0 \longrightarrow S(2) \xrightarrow{f'} S(1)\oplus S(2) \xrightarrow{g'} S(1) \longrightarrow 0.$

w/ $f' = (f_1', f_2') = (0,1)$ & $g' = (g_1', g_2') = (1,0)$

is a short exact seq. that splits. Since $\exists h \in \mathrm{Hom}(S(1)\oplus S(2), S(2))$

i.e. $h : (S(1)\oplus S(2)) \xrightarrow{(0,1)} S(2) \Rightarrow h \circ f' = 1_{S(2)}$

Prop. Let $0 \longrightarrow L \xrightarrow{f} M \xrightarrow{g} N \longrightarrow 0$ be a short exact seq. in rep Q. Then,

(a) f is a section iff g is a retraction

(b) If f is a section, then $\operatorname{im} f = \ker g$ is a direct summand of M.

Pf (sketch).

(a) $\Rightarrow$ f a section $\Rightarrow \exists h \in \mathrm{Hom}(M, L)$ s.t. $h \circ f = 1_L$

Define $h' : N \to M$ s.t. let $n \in N$ $\quad g \twoheadrightarrow \Rightarrow \exists m \in M$ s.t. $g(m) = n$,

$\hookrightarrow h'(n) = m - f\circ h(m)$. Show h' is well-defined.

Show h' is a morphism. $L = (L_i, \varphi_\alpha)$, $M = (M_i, \varphi_\alpha')$, $N = (N_i, \varphi_\alpha'')$

$$\begin{tikzcd}
L_i \arrow[r, "\varphi_\alpha"] \arrow[d, "f_i"] & L_j \arrow[d, "f_j"'] \\
M_i \arrow[r, "\varphi_\alpha'"] \arrow[u, bend left, "h_i"] \arrow[d, "g_i"] & M_j \arrow[u, bend right, "h_j"'] \arrow[d, "g_j"'] \\
N_i \arrow[r, "\varphi_\alpha''"] \arrow[u, bend left, "h_i'"] & N_j \arrow[u, bend right, "h_j'"']
\end{tikzcd}$$

NTS commutation wrt h' (using commutativity of f, g, h)

$\vdots \Rightarrow$ Seq. exact $\Rightarrow gh' = 1_N$.

Pf (sketch cont.)

(a) $\Leftarrow$ g retraction $\Rightarrow \exists h' \in \mathrm{Hom}(N, M)$ s.t. $g \circ h' = 1_N$.

Define $h: M \to L$ s.t. let $m \in M$ then,

$m - h'(g(m)) \in \ker g = \operatorname{im} f \Rightarrow \exists! \ell \in L$ s.t. $f(\ell) = m - h'g(m)$

$\hookrightarrow h(m) = \ell$.

Follows that $h \circ f = 1_L$.

Show h is a morphism $\in \operatorname{rep} Q$.

$\vdots$

(b) let $h' \in \mathrm{Hom}(N, M)$ s.t. $gh' = 1_N$. Let $m = (m_i) \in M$.

Then $m_i = h_i' g_i(m_i) + (m_i - h_i' g_i(m_i))$

where $h_i' g_i(m_i) \in \operatorname{im} h_i'$ & $(m_i - h_i' g_i(m_i)) \in \ker g_i$.

$g \circ h' = 1_N \Rightarrow \operatorname{im} h_i' \cap \ker g_i = \{0\}$

$\Rightarrow \forall M_i \in M$ we have $M_i = \operatorname{im} h_i' \oplus \ker g_i$ (vector spaces).

NTS $\varphi_\alpha \in M$ are the maps of $\operatorname{im} h_i' \oplus \ker g_i$.

$$\hookrightarrow \quad \varphi_\alpha' = \begin{bmatrix} \varphi_\alpha'|_{\operatorname{im} h_i'} & 0 \\ 0 & \varphi_\alpha'|_{\ker g_i} \end{bmatrix}$$

$\square$

... we show $M = \operatorname{im} h' \oplus \ker g$.

Cor. If $0 \to L \xrightarrow{f} M \xrightarrow{g} N \to 0$ is split exact Then $M \simeq L \oplus N$.

Pf. $f \hookrightarrow \Rightarrow L \simeq f(L) \simeq \ker g$ & $g \twoheadrightarrow \overset{1^{st} \text{ iso. thm}}{\Rightarrow} N \simeq M/\ker g$.

$\Rightarrow M \simeq L \oplus N$ by above prop.

$\square$

1.4 Hom Functors

<u>Defn.</u> Let $\mathcal{C}$ and $\mathcal{C}'$ be two k-categories. A <u>covariant functor</u> $F: \mathcal{C} \to \mathcal{C}'$ is a mapping s.t.

- $(X \in \mathcal{C}) \longmapsto (F(X) \in \mathcal{C}')$
- $(f: X \to Y) \in \mathcal{C} \longmapsto (F(f): F(X) \to F(Y)) \in \mathcal{C}'$ s.t.

$F(1_X) = 1_{F(X)}$ and $F(g \circ f) = F(g) \circ F(f)$.

A <u>contravariant functor</u> $F: \mathcal{C} \to \mathcal{C}'$ is a mapping s.t.

- $(X \in \mathcal{C}) \longmapsto (F(X) \in \mathcal{C}')$
- $(f: X \to Y) \in \mathcal{C} \longmapsto (F(f): F(Y) \to F(X)) \in \mathcal{C}'$ s.t.

$F(1_X) = 1_{F(X)}$ and $F(g \circ f) = F(f) \circ F(g)$

Fix $X \in \mathcal{C}$, two important functors are the Hom functors: $\mathrm{Hom}(X, -)$ & $\mathrm{Hom}(-, X)$.

$\mathrm{Hom}(X, -)$ is the covariant functor from $\mathcal{C}$ to Cat_k sends $Y \in \mathcal{C}$ to $\mathrm{Hom}(X, Y) \in \mathrm{Cat}_k$ and a morphism $f: Y \to Z \in \mathcal{C}$ to $f_*: \mathrm{Hom}(X, Y) \to \mathrm{Hom}(X, Z)$ s.t. $f_*(g) = f \circ g$:

$$\begin{tikzcd} X \arrow[d, "g"'] \arrow[dr, "f \circ g"] & \\ Y \arrow[r, "f"'] & Z \end{tikzcd}$$

This is the <u>push forward of</u> f.

$Hom(-,X)$ is the contravariant functor from $\mathcal{C}$ to Cat_k.

Sends $Y \in \mathcal{C}$ to $Hom(Y,X) \in Cat_k$ and a morphism

$f: Y \to Z \in \mathcal{C}$ to $f^*: Hom(Z,X) \to Hom(Y,X)$

s.t. $f^*(g) = g \circ f$:

$$\begin{tikzcd} Y \arrow[r, "f"] \arrow[dr, "g\circ f"'] & Z \arrow[d, "g"] \\ & X \end{tikzcd}$$

The map f^* is called the <u>pull back</u> of f.

Apply these notions to $\mathcal{C} = \text{rep}\,Q$.

<u>Thm.</u> Let Q be a quiver & $0 \to L \xrightarrow{f} M \xrightarrow{g} N$ a sequence in rep Q. Then the sequence is exact iff for every representation $X \in \text{rep}\,Q$, the following seq. is exact.

$$0 \longrightarrow Hom(X,L) \xrightarrow{f_*} Hom(X,M) \xrightarrow{g_*} Hom(X,N)$$

<u>Pf. (sketch).</u>

($\Rightarrow$) Show f_* injective, sps $u \in Hom(X,L)$ s.t. $f \circ u = f_*(u) = 0$

f injective $\Rightarrow u = 0 \Rightarrow f_*$ injective.

Show $\text{im} f_* = \ker g_*$. Let $u \in Hom(X,L)$ Then $g_* f_*(u) = g \circ f \circ u = 0$

Since $g \circ f = 0 \Rightarrow g_* f_* = 0 \Rightarrow \text{im} f_* \subset \ker g_*$.

Let $v \in Hom(X,M)$ s.t. $v \in \ker g_*$ then $0 = g_*(v) = g \circ v$

by universal property of kernel of g v factors through $f \Rightarrow \exists\, u \in Hom(X,L)$

s.t. $v = f \circ u = f_*(u) \Rightarrow v \in \text{im} f_* \Rightarrow \ker g_* \subset \text{im} f_*$

Thus $\text{im} f_* = \ker g_*$

Pf (sketch cont.)

($\Leftarrow$) Show f is injective, let $X = \ker f$ and let $i: X \hookrightarrow L$ be the inclusion morphism. Then $0 = f \circ i = f_*(i)$, since f_* injective
$\Rightarrow i = 0 \xrightarrow[i \text{ injective}]{} X = 0 \Rightarrow f$ is injective.

Show $\operatorname{im} f = \ker g$, let $X = L$ then $0 = g_* f_*(1_L) = g \circ f \circ 1_L = g \circ f \in$
Then $\operatorname{im} f \subset \ker g$.
Let $X = \ker g$ and $i: X \hookrightarrow M$ the inclusion morphism. Then
$0 = g \circ i = g_*(i) \Rightarrow i \in \ker g_* = \operatorname{im} f_* \Rightarrow \exists\, v \in \operatorname{Hom}(X, L)$
s.t. $i = f_*(v) = f \circ v$ then $\ker g = i(X) \subset \operatorname{im} f$.
Thus $\operatorname{im} f = \ker g$. $\square$

Cor. A sequence $0 \to L \xrightarrow{f} M \xrightarrow{g} N \to 0$ in rep$Q$ is split exact iff for every $X \in$ rep$Q$ the following seq. is exact.

$$0 \to \operatorname{Hom}(X, L) \xrightarrow{f_*} \operatorname{Hom}(X, M) \xrightarrow{g_*} \operatorname{Hom}(X, N) \to 0$$

Pf.

($\Rightarrow$) Suffices to show g_* surjective. Sps the seq. is exact (split) then g retraction $\exists h' \in \operatorname{Hom}(N, M)$ s.t. $gh' = 1_N$. For any $v \in \operatorname{Hom}(X, N)$ we have $h'v \in \operatorname{Hom}(X, M)$ and $g_*(h'v) = gh'v = 1_N v = v \Rightarrow g_*$ is surjective.

($\Leftarrow$) Sps for $X \in$ rep$Q$ the seq. is exact. then $0 \to L \xrightarrow{f} M \xrightarrow{g} N$ is exact.
Let $X = N$ & g surjective $\Rightarrow \exists\, h \in \operatorname{Hom}(N, M)$ s.t. $1_N = g_*(h) = gh$

$\Rightarrow$ 1) g is surjective $\Rightarrow$ the seq. is exact, and
2) g is a retraction $\Rightarrow$ the seq. splits. $\square$

Dual versions exist for the $Hom(-,X)$ functor. Note the order of reps L, M, N is reversed since $Hom(-,X)$ is contravariant.

<u>Thm.</u> Let Q be a quiver & $L \xrightarrow{f} M \xrightarrow{g} N \longrightarrow 0$ a seq. in rep Q. Then this seq. is exact iff $\forall X \in$ rep Q the following seq. is exact.

$$0 \longrightarrow Hom(N,X) \xrightarrow{g^*} Hom(M,X) \xrightarrow{f^*} Hom(L,X)$$

<u>Cor.</u> A seq. $0 \longrightarrow L \xrightarrow{f} M \xrightarrow{g} N \longrightarrow 0 \in$ rep Q is split exact iff $\forall X \in$ rep Q the following seq. is exact.

$$0 \longrightarrow Hom(N,X) \xrightarrow{g^*} Hom(M,X) \xrightarrow{f^*} Hom(L,X) \longrightarrow 0$$

<u>RMK</u> If $0 \longrightarrow L \xrightarrow{f} M \xrightarrow{g} N \longrightarrow 0$ does not split then f^* & g_* are not always surjective. We can extend the exact sequences of Thms. to the right by use of the <u>extension functors</u> $Ext^i(X,-)$ & $Ext^i(-,X)$ (§2.4)

1.5 First Ex. of Auslander-Reiten Quivers.

Recall goal of rep. theory of quivers is to study reps, morphisms, exact seq. in Rep Q for a given quiver Q.

The Auslander-Reiten quiver is an approximation for rep Q.

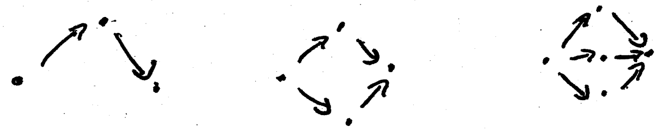
Let Q be a quiver the AR-quiver is a new quiver Γ_Q where

- The vertices are the isoclass of indecomposable reps.
- The arrows are the <u>irreducible morphisms</u>

Build any $M \in$ rep Q from the vertices of Γ_Q (Building blocks of reps.)

Build <u>most</u> $f: M \to N \in$ rep Q from the arrows of Γ_Q (Building blocks of morphism of reps)

Studying short exact seq. comes from gluing together the almost split sequences, we can construct most short exact seq. this way. These almost split seq. show in the AR-quiver as meshes.



Notation. Let $Q_0 = \{1, 2, \ldots, n\}$, M an indecomposable $\in \operatorname{rep} Q$ let $\underline{\dim}\, M = (d_1, d_2, \ldots, d_n)$. Configure the rep. M so digit i appears d_i times. If $\alpha : i \to j$ is s.t. $\varphi_\alpha : M_i \to M_j$ is non-zero the digit i is placed above digit j.

Ex. Let $Q : 1 \longrightarrow 2 \longleftarrow 3$. There are six indecom. reps

$S(2)$: $0 \longrightarrow k \longleftarrow 0$

$P(1)$: $k \xrightarrow{1} k \longleftarrow 0$

$P(3)$: $0 \longrightarrow k \xleftarrow{1} k$

$I(2)$: $k \xrightarrow{1} k \xleftarrow{1} k$

$S(1)$: $k \longrightarrow 0 \longleftarrow 0$

$S(3)$: $0 \longrightarrow 0 \longleftarrow k$

where in our notation:

$$S(2) = 2,\quad P(1) = \begin{smallmatrix} 1 \\ 2 \end{smallmatrix},\quad P(3) = \begin{smallmatrix} 3 \\ 2 \end{smallmatrix},\quad I(2) = \begin{smallmatrix} 1 & 3 \\ & 2 \end{smallmatrix},\quad S(1) = 1,\quad S(3) = 3$$

w/ three almost split seq.

$$0 \longrightarrow 2 \longrightarrow \begin{smallmatrix} 1 \\ 2 \end{smallmatrix} \oplus \begin{smallmatrix} 3 \\ 2 \end{smallmatrix} \longrightarrow \begin{smallmatrix} 1 & 3 \\ & 2 \end{smallmatrix} \longrightarrow 0, \qquad 0 \longrightarrow \begin{smallmatrix} 1 \\ 2 \end{smallmatrix} \longrightarrow \begin{smallmatrix} 1 & 3 \\ & 2 \end{smallmatrix} \longrightarrow 3 \longrightarrow 0,$$

$$0 \longrightarrow \begin{smallmatrix} 3 \\ 2 \end{smallmatrix} \longrightarrow \begin{smallmatrix} 1 & 3 \\ & 2 \end{smallmatrix} \longrightarrow 1 \longrightarrow 0$$

So Γ_Q is

$$\begin{tikzcd}
 & \begin{smallmatrix} 3 \\ 2 \end{smallmatrix} \arrow[dr] & & 1 \\
2 \arrow[ur] \arrow[dr] & & \begin{smallmatrix} 1 & 3 \\ & 2 \end{smallmatrix} \arrow[ur] \arrow[dr] & \\
 & \begin{smallmatrix} 1 \\ 2 \end{smallmatrix} \arrow[ur] & & 3
\end{tikzcd}$$

(verify in GAP using QPA wip.)

Ch. 2 Projective & Injective Reps.

A rep $P \in \operatorname{rep} Q$ is called <u>projective</u> if $\operatorname{Hom}(P, -)$ maps surjective morphisms to surjective morphisms.

A rep $I \in \operatorname{rep} Q$ is called <u>injective</u> if $\operatorname{Hom}(-, I)$ maps injective morphisms to injective morphisms.

For any $M \in \operatorname{rep} Q$; $\exists\, P_0, I_0 \in \operatorname{rep} Q$ s.t. $\exists\, p_0, i_0$

$$p_0 : P_0 \twoheadrightarrow M \quad \& \quad i_0 : I_0 \hookrightarrow M$$

A <u>projective resolution</u> is of the form $\cdots \to P_2 \xrightarrow{p_2} P_1 \xrightarrow{p_1} P_0 \xrightarrow{p_0} M \to 0$

<u>injective resolution</u> is $0 \to M \xrightarrow{i_0} I_0 \xrightarrow{i_1} I_1 \xrightarrow{i_2} I_2 \to \cdots$

Let Q be finite quiver w/o orientated cycles every representation has proj / inj resolution of the form.

$$0 \to P_1 \to P_0 \to M \to 0 \qquad 0 \to M \to I_0 \to I_1 \to 0.$$

<u>Goal</u>: Use the Auslander-Reiten translation τ in conjunction w/ the Nakayama functor to bridge Projective res. to Injective resolutions.

<u>Defn.</u> Let Q be a quiver, $i, j \in Q_0$. A <u>path</u> c from i to j of length ℓ is a seq. $c = (i | \alpha_1, \ldots, \alpha_\ell | j)$ s.t. $s(\alpha_1) = i$
$s(\alpha_h) = t(\alpha_{h-1})$, & $t(\alpha_\ell) = j$.

$$i \qquad\qquad i \circlearrowleft^{\alpha} \qquad\qquad \begin{tikzcd} i \arrow[r, "\alpha_1"] & \bullet \arrow[r, "\alpha_2"] & \cdots \arrow[r, "\alpha_{\ell-1}"] & \bullet \arrow[lll, bend left, "\alpha_\ell"] \end{tikzcd}$$

$c = (i||i)$ Constant/Lazy path

$c = (i|\alpha|i)$ Loop

$c = (i|\alpha_1, \ldots, \alpha_\ell|i)$ Orientated Cycle.

2.1 Simple, Proj., & Inj. Representations.

Let Q be a quiver w/o orientated cycles.

<u>Defn.</u> Let $i \in Q_0$ be a vertex. We have the following representations.

a) $S(i)$ dimension 1 at vertex i, 0 otherwise

$$S(i) = (S(i)_j, \varphi_\alpha); \quad S(i)_j = \begin{cases} \mathbb{K} & i = j \\ 0 & i \neq j \end{cases} \quad \varphi_\alpha = 0$$

b) $P(i) = (P(i)_j, \varphi_\alpha)$ where $P(i)_j$ is the $\mathbb{K}$-vector space of basis as the set of all paths from i to j. If $j \xrightarrow{\alpha} l$ is an arrow in Q then $\varphi_\alpha : P(i)_j \to P(i)_l$ is the map defined on the basis by composing paths from i to j. That is, the arrow α induces an injective map of bases

$$\begin{array}{ccc} \text{basis of } P(i)_j & \longrightarrow & \text{basis of } P(i)_l \\ c = (i \mid \beta_1, \ldots, \beta_s \mid j) & \longmapsto & c\alpha = (i \mid \beta_1, \ldots, \beta_s, \alpha \mid l) \end{array}$$

And $\varphi_\alpha : \sum_c \lambda_c c \longmapsto \sum_c \lambda_c c\alpha$.

c) $I(i) = (I(i)_j, \varphi_\alpha)$ where $I(i)_j$ is the $\mathbb{K}$-vector space w/ basis as the set of all paths from $j \to i$. If $j \xrightarrow{\alpha} l$ is an arrow in Q then $\varphi_\alpha : I(i)_j \to I(i)_l$ is the map defined on the basis by composing paths from j to i. ~~which~~ That is, the arrow α induces a surjective map of bases f:

$$\begin{array}{ccc} \text{basis of } I(i)_j & \xrightarrow{\ f\ } & \text{basis of } I(i)_l \\ c = (j \mid \beta_1, \ldots, \beta_s \mid i) & \longmapsto & \begin{cases} (l \mid \beta_2, \ldots, \beta_s \mid i) & \text{if } \beta_1 = \alpha \\ 0 & \text{otherwise} \end{cases} \end{array}$$

And $\varphi_\alpha : \sum_c \lambda_c c \longmapsto \sum_c \lambda_c f(c)$.

The reps. $S(i)$, $P(i)$, & $I(i)$ are called <u>simple</u>, <u>projective</u>, & <u>injective</u> representations resp.

If Q has orientated cycle(s) $\Longrightarrow \exists\, i \in Q_0$ s.t. $P(i)$ is infinite dimensional $\Longrightarrow P(i) \notin \operatorname{rep} Q$.

$\ast$ <u>Rmk.</u> Let $P(i) \in \operatorname{rep} Q$ & c a path starting at i. $c = (i|\alpha_1, \ldots, \alpha_\ell|j)$.

Then we have the map.

$$\varphi_c : P(i)_i \longrightarrow P(i)_j \qquad \varphi_c = \varphi_{\alpha_\ell} \cdots \varphi_{\alpha_1}$$

is a composistion of the maps in the rep. $P(i)$ along c. If e_i is the constant path of i then $\varphi_c(e_i) = c$.

$S(i) = P(i) \Longleftrightarrow i$ is a <u>sink</u> in Q,

$S(i) = I(i) \Longleftrightarrow i$ is a <u>source</u> in Q.

<u>Ex.</u> Let Q be

$$\begin{tikzcd} 1 \arrow[r] & 2 & 3 \arrow[l] \arrow[d] & 4 \arrow[l] \\ & & 5 & \end{tikzcd}$$

Then,

$$S(3) = [0,0,1,0,0] \simeq \begin{tikzcd} 0 \arrow[r] & 0 & \mathbb{K} \arrow[l] \arrow[d] & 0 \arrow[l] \\ & & 0 & \end{tikzcd}$$

$$P(3) = [0,1,1,0,1] \simeq \begin{tikzcd} 0 \arrow[r] & \mathbb{K} & \mathbb{K} \arrow[l, "1"'] \arrow[d, "1"] & 0 \arrow[l] \\ & & \mathbb{K} & \end{tikzcd}$$

$$I(3) = [0,0,1,1,0] \simeq \begin{tikzcd} 0 \arrow[r] & 0 & \mathbb{K} \arrow[l] \arrow[d] & \mathbb{K} \arrow[l, "1"'] \\ & & 0 & \end{tikzcd}$$

Following prop. shows $P(i)$ is a <u>projective</u> object wrt Category theory. ($\mathrm{Hom}(P_0, -)$ maps surjective to surjective, <u>push forward</u>)

<u>Prop.</u> Let $g: M \to N$ be a surjective morphism of reps M, N of Q. Let $P(i)$ be the projective rep. at $i \in Q_0$. Then the map

$$g_*: \mathrm{Hom}(P(i), M) \longrightarrow \mathrm{Hom}(P(i), N)$$

is surjective. That is if $f: P(i) \to N$, $\exists\, h$ s.t. $h: P(i) \to M$ and the diagram

$$\begin{tikzcd}
 & P(i) \arrow[dl, "h"'] \arrow[d, "f"] & \\
M \arrow[r, "g"'] & N \arrow[r] & 0
\end{tikzcd}$$

commutes, $f = g \circ h = g_*(h)$.

<u>Cor.</u> If P is projective then $0 \to L \to M \xrightarrow{g} P \to 0$ splits.

<u>Pf.</u> Let $f = 1_P$ then we get the commutative diagram.

$$\begin{tikzcd}
 & P \arrow[dl, "h"'] \arrow[d, no head] & \\
M \arrow[r, "g"'] & P \arrow[r] & 0
\end{tikzcd}$$

$\Rightarrow 1 = g \circ h$ & g is a retraction. $\square$

Dual statements of above Prop & Cor. hold for <u>injective</u> objects wrt Categories ($\mathrm{Hom}(-, I)$ maps injective maps to surjection, <u>pull back</u>)

<u>Simple</u> objects in a Category are nonzero w/ no proper subobjects.

Prop. Let $I, I', P, P' \in \operatorname{rep} Q$. Then

1) $P \oplus P'$ projective $\iff P, P'$ projective
2) $I \oplus I'$ injective $\iff I, I'$ injective

Note: This prop. holds not just for rep$Q$. But for any additive category.

Pf. of 1) ($\Rightarrow$) Let $g: M \to N$ be surjective in rep$Q$ & $f: P \to N$ be any morphism. Consider the diagram

$$\begin{tikzcd}
& P \oplus P' \arrow[ddl, "\exists h"'] \arrow[d, "pr_1"', shift right] \\
& P \arrow[u, "i_1"', bend right] \arrow[d, "f"] \arrow[dl, dashed, "h'"'] \\
M \arrow[r, "g"'] & N \arrow[r] & 0
\end{tikzcd}$$

w/ pr_1 proj. on the summand

i_1 cannonical injection

Clearly, $pr_1 \circ i_1 = 1_P$

Since $P \oplus P'$ projective, $\exists\, h: P \oplus P' \to M$ s.t. $gh = f\,pr_1$
$\Rightarrow g h i_1 = f\,pr_1 i_1 = f 1_P = f$. Define $h': P \to M$ as
$h' = h i_1$ so $gh' = f$. Therefore P is projective. Similar arg. holds for P'.

($\Leftarrow$) Let $g: M \to N$ be surjective & $f: P \oplus P' \to N$ be any morphism.
Consider the diagram

$$\begin{tikzcd}
& P \arrow[d, "i_1"] \arrow[ddl, "\exists h"'] \\
& P \oplus P' \arrow[d, "f"] \arrow[dl, dashed, "h"'] \\
M \arrow[r, "g"'] & N \arrow[r] & 0
\end{tikzcd}$$

w/ i_1 cannonical injection

~~Being~~ P proj. $\Rightarrow \exists\, h_1$ s.t.
$gh_1 = f i_1$ & by symmetry $\exists h_2$ s.t.
$gh_2 = f i_2$

Define $h = (h_1, h_2): P \oplus P' \to M$ by $h(p + p') = h_1(p) + h_2(p')$. Then

$$gh(p+p') = gh_1(p) + gh_2(p) = f i_1(p) + f i_2(p') = f(p+p')$$

Therefore $P \oplus P'$ is projective. $\square$

By previous prop. if we know all indecomposable proj. (resp. inj) reps, then we know all proj. (resp. inj) reps.

<u>Prop.</u> The representations $S(i)$, $P(i)$, $I(i)$ are indecomposable.

<u>Pf.</u> $S(i)$ follows immediatley. We show $P(i)$ as $I(i)$ is similar idea.

Q no orientated cycles $\Rightarrow$ $P(i)_i = \mathbb{k}$.

Sps $P(i) = M \oplus N$ for $M, N \in \text{rep } Q$. WLOG sps $P(i)_i = M_i$ & $N_i = 0$

Let l be a vertex in Q s.t. $N_l \neq 0$. $\Rightarrow$ $P(i)_l$ has a basis of paths from i to l, let $c = (i|\beta_1, \dots, \beta_s|l)$ be a path.

Let $\varphi_c = \varphi_{\beta_s} \cdots \varphi_{\beta_1}$ be comp. of linear maps of the rep. $P(i)$ along c.

Since $P(i) = M \oplus N$ then $\varphi_c : M_i \oplus 0 \longrightarrow M_l \oplus N_l$

Sends unique basis element e_i of M_i to element $\varphi_c(e_i)$ of M_l.

However By previous rmk $\varphi_c(e_i) = c$ $\Rightarrow$ every basis element c of $P(i)_l$ is in M_l, which is a contradiction $\square$

<u>Prop.</u> A rep. of Q is simple iff its isomorphic to $S(i)$ for some i.

<u>Pf.</u> ($\Leftarrow$) Immediate. ($\Rightarrow$) Let $i \in Q_0$ s.t. $M_i \neq 0$ & $M_j = 0$ if $\exists\, i \xrightarrow{\alpha} j$ (Q has no orientated cycles) Choose $f_i : S(i)_i \simeq \mathbb{k} \to M_i$ & extend to $f : S(i) \to M$ if $f_j = 0$ if $i \neq j$. Then the diagram commutes for all arrows

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & S(i)_i \arrow[r] \arrow[d, "f_i"] & 0 \arrow[d] \\
M_l \arrow[r, "\varphi_\alpha"] & M_i \arrow[r, "\varphi_\beta"] & 0
\end{tikzcd}$$

$\Rightarrow$ $S(i)$ is subrep. of M

$\Rightarrow$ $M \simeq S(i)$ or M is not simple $\square$

<u>Thm.</u> Let $M \in \text{rep}\,Q$. For any $i \in Q_0$ we have
$$\text{Hom}(P(i), M) \simeq M_i.$$

<u>Pf.</u> Let $e_i = (i \| i) \Rightarrow \text{Span}(\{e_i\}) = P(i)_i$. Define
$$\phi : \text{Hom}(P(i), M) \longmapsto M_i \quad \text{s.t.} \quad \phi : f = (f_j)_{j \in Q_0} \longmapsto f_i(e_i).$$

if f is a morphism from $P(i)$ to M then f_i is a liner map from $P(i)_i$ to M_i
so ϕ is well-defined since $e_i \in P(i)_i$.

NTS ϕ isomorphism.

↳ <u>Injectivity</u>: If $0 = \phi(f) = f_i(e_i) \Rightarrow f_i$ sends e_i to zero & is the zero map. Can show $f_j : P(i)_j \longrightarrow M_j$ is zero map for any vertex j.
$P(i) \Rightarrow \text{Span}(\{c \mid c(i|\cdots|j)\}) = P(i)_j$. Let $c = (i|\alpha_1 \dots \alpha_t|j)$
be a basis elem. of $P(i)_j$. It follows that $\varphi_c(e_i) = c$. Then
f is a morphism $\Rightarrow f_j \varphi_c = \varphi'_c f_i$ since $f_i(e_i) = 0 \Rightarrow f_j : c \longmapsto 0$.
c arbitrary shows $f_j = 0$.

↳ <u>Surjectivity</u>: Let $m_i \in M_i$. Construct $f : P(i) \rightarrow M$ s.t. $f_i(e_i) = m_i$.
Fix $f_i : P(i)_i \longrightarrow M_i$ s.t. $f_i(e_i) = m_i$. Since $\{e_i\}$ is a basis of $P(i)_i$
this defines f_i as liner map in a unique way. Extend f_i to a
morphism f by <u>following the paths in Q</u>. If $c = (i|\cdots|j)$ then
$f_j(c) = \varphi'_c(m_i)$. It follows that f is a morphism of reps &
$f \in \text{Hom}(P(i), M)$ & $\phi(f) = m_i$ so ϕ surjective.

$\square$

ϕ inj. & surj $\Rightarrow \text{Hom}(P(i), M) \simeq M_i$

Let $i, j \in Q_0$ then:

1) $\mathrm{Hom}(P(i), P(j))$ has basis consisting of all paths from j to i in Q. Moreover,

$$\mathrm{End}(P(i)) = \mathrm{Hom}(P(i), P(i)) \simeq \mathbb{K}.$$

2) If $A = \bigoplus_{i \in Q_0} P(i)$ then $\mathrm{End}(A) = \mathrm{Hom}(A, A)$ has a basis of all paths in Q.

The <u>Path Algebra</u> of Q is isomorphic to $\mathrm{End}(A)$ as both vector space & Algebra.

<u>Pf. of 1)</u>. $\mathrm{Hom}(P(i), P(j)) \simeq P(j)_i \Rightarrow$ has basis of all paths from j to i. Q has no orientated cycles $\Rightarrow \mathrm{End}\,P(i))$ has dim of 1. $\Rightarrow \mathrm{End}(P(i)) \simeq \mathbb{K}$.

<u>Cor.</u> $P(j)$ simple iff $\mathrm{Hom}(P(i), P(j)) = 0 \ \forall j \neq i$.

<u>Pf.</u> $P(j)$ simple iff j is a sink, i.e. no paths from j to any vertex i. Follows from Above. $\square$

(2.2) Proj. Res & Radicals of Projectives

Defn. A projective resolution is an exact seq of the form

$$\cdots \longrightarrow P_2 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

where the P_i are projective. An injective resolution is an exact seq.

$$0 \longrightarrow M \longrightarrow I_0 \longrightarrow I_1 \longrightarrow I_2 \longrightarrow \cdots$$

w/ the I_i as injective.

Thm. Let $M \in \operatorname{rep} Q$. There exists projective / injective resolutions.

(1) $\quad 0 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0.$

(2) $\quad 0 \longrightarrow M \longrightarrow I_0 \longrightarrow I_1 \longrightarrow 0.$

Pf. (projective res. only):

Goal: Construct the standard projective resolution.

Let $M = (M_i, \varphi_\alpha)$ let $d_i = \dim M_i$. Define

$$P_1 = \bigoplus_{\alpha \in Q_1} d_{s(\alpha)} P(t(\alpha)) \qquad P_0 = \bigoplus_{i \in Q_0} d_i P(i)$$

where $d_i P(i)$ is $\underbrace{P(i) \times \cdots \times P(i)}_{d_i \text{ times}}$.

For every M_i we have $P(i)$ w/ $\dim M_i$ copies of $P(i)$ in P_0

Naturally $g: P_0 \longrightarrow M$ sends $d_i P(i)$, d_i copies of the constant path e_i in P_0 to a basis of M_i.

Each copy of $P(i)$ the Ker g contains a copy of $P(t(\alpha))$ for all α s.t. $s(\alpha) = i$.

Pf. (continued)

That is we clearly have $d_{s(\alpha)}$ copies of $P(t(\alpha))$ in $\ker g$. So the defn. of P_1 follows.

Next: Define morphisms of the projective resolution.
For each $M_i \in M$ let $\{m_{i1}, \dots, m_{id_i}\}$ be it's basis. Then,

$$\mathcal{B}'' := \{ m_{ij} \mid i \in Q_0, j = 1, 2, \dots, d_i \}$$

is a basis for M. The set,

$$\mathcal{B} := \{ c_{ij} \mid i \in Q_0, \ c_i = (i|\dots|i), \ j = 1, 2, \dots, d_i \}$$

is the standard basis for P_0. The set,

$$\mathcal{B}' := \{ b_{\alpha j} \mid \alpha \in Q_1, \ b_\alpha = (t(\alpha)|\dots|j), \ j = 1, \dots, d_i \}$$

is a basis for P_1. Define g on the basis $\mathcal{B}$ by

$$g(c_{ij}) = \varphi_{c_i}(m_{ij}) \in M_{t(c_i)}$$

extend g (linearly) to P_1. Define f on the basis $\mathcal{B}'$ by

$$f(b_{\alpha j}) = (\alpha b_\alpha)_j - b_\alpha^M$$

where $\alpha b_\alpha = (s(\alpha)|\dots|t(b_\alpha))$ by comp. of α & b_α and $b_\alpha^M = \sum_{l=1}^{d_{t(\alpha)}} \Theta_l b_{\alpha l}$, Θ_l are the scalars of $\varphi_\alpha(m_{s(\alpha)j})$ in its basis.

Thus,

$$\varphi_\alpha(m_{s(\alpha)j}) = \sum_{l=1}^{d_{t(\alpha)}} \Theta_l m_{t(\alpha)l}.$$

Pf. (continued)

Now we show the following is exact.

$$0 \longrightarrow \bigoplus_{\alpha \in Q_1} d_{s(\alpha)} P(t(\alpha)) \xrightarrow{f} \bigoplus_{i \in Q_0} d_i P(i) \xrightarrow{g} M \longrightarrow 0$$

g surjective

Any basis vector m_{ij} of M we have $m_{ij} = g(e_{ij})$ where e_i is constant path at vertex i.

$\hookrightarrow \ker g \supset \operatorname{im} f$: suffices to show $g \circ f(b_{\alpha j}) = 0$, $b_{\alpha j} \in B'$

$$g(f(b_{\alpha j})) = g((\alpha b_\alpha)_j - b_\alpha^M) = \varphi_{\alpha b_\alpha}(m_{s(\alpha)j}) - \varphi_{b_\alpha}\left(\sum_l \theta_l m_{t(\alpha)l}\right)$$

$$= \varphi_{b_\alpha}\left(\varphi_\alpha(m_{s(\alpha)j}) - \sum_l \theta_l m_{t(\alpha)l}\right)$$

$$= \varphi_{b_\alpha}(0) = 0 \quad \text{[Follows from defns above]}$$

$\hookrightarrow \ker g \subset \operatorname{im} f$: Any $x \in \bigoplus_{i \in Q_0} d_i P(i)$ is a linear combination of basis B:

$$x = \sum_{c_{ij} \in B} \lambda_{c_{ij}} c_{ij} = x_0 + \sum_{c_{ij} \in B \setminus B_0} \lambda_{c_{ij}} c_{ij}$$

w/ B_0 the subset of B w/ constant paths

$$B_0 = \{ e_{ij} \mid i \in Q_0, \ j = 1, \dots, d_i \}$$

and $x_0 = \sum_{e_{ij} \in B_0} \lambda_{e_{ij}} e_{ij}$.

Pf. (continued)

$\hookrightarrow \ker g \subset \operatorname{im} f$ (cont.):

Any nonconstant path is the product of an arrow and another path;

$$x = x_0 + \sum_{c_{ij} : c_i = \alpha b_\alpha} \lambda_{c_{ij}} (\alpha b_\alpha);$$

By f we get:

$$x = x_0 + \sum_{c_{ij} : c_i = \alpha b_\alpha} \lambda_{c_{ij}} f(b_{\alpha_j}) + \lambda_{c_{ij}} b_\alpha^M \qquad (\star)$$

Let $x_1 = x_0 + \sum_{c_i = \alpha b_\alpha} \lambda_{c_{ij}} b_\alpha^M$. Moreover, $x - x_1 \in \operatorname{im} f$

Define the degree of a lin. combination of paths to be the length of the longest path w/ nonzero coefs. $\deg x_1 < \deg x$ & $\deg x_0 = 0$.

NTS $x \in \ker g \Rightarrow x \in \operatorname{im} f$. From $(\star)$ & $g \circ f = 0$ we have

$$0 = g(x) = g(x_1)$$

$\Rightarrow x_1 \in \ker g$, $\deg x_1 < \deg x$, & $x - x_1 \in \operatorname{im} f$

Repeat this for all x_i until we get to an x_h s.t. $x_h \in \ker g$, $x - x_h \in \operatorname{im} f$, & $\deg x_h = 0 \Rightarrow x_h$ is comb. of constant paths. $\Rightarrow x_h = 0$ & $x \in \operatorname{im} f$. So $\ker g \subset \operatorname{im} f$.

Pf. (Continued)

f injective

Sps $0 = f(\sum \lambda_{b_{\alpha h}} b_{\alpha h}) = \sum \lambda_{b_{\alpha h}} \left((\alpha b_\alpha)_h - b_\alpha^M \right)$.

Then, $\sum \lambda_{b_{\alpha h}} (\alpha b_{\alpha h})^g = \sum \lambda_{b_{\alpha h}} b_\alpha^M = \sum \lambda_{b_{\alpha h}} \sum \theta_l b_{\alpha l}$

Let i_0 be a source in M, note that i_0 exists since Q has no orientated cycles.

Each path b_α starts at the endpoint of arrow α, none of b_α go through $i_0 \Rightarrow \lambda_{b_{\alpha h}} = 0$, $\forall \alpha \in Q_1$ s.t. $s(\alpha) = i_0$.

Let i_1 be a source in $M \setminus i_0$ ...

We continue this until we get that every $\lambda_{b_{\alpha h}} = 0$ (since Q finite) $\Rightarrow f$ is injective.

Since f injective & g surjective we get the fact that the projective resolution is exact.

$\square$

Ex.

Let $Q := 1 \longrightarrow 2 \longleftarrow 3$ w/ reps

$M = S(3) := 0 \longrightarrow 0 \longleftarrow \mathbb{k}$, $M' := \mathbb{k} \longrightarrow \mathbb{k} \longleftarrow \mathbb{k}$

in other notation $M = 3$, $M' = \begin{smallmatrix}1&&3\\&2&\end{smallmatrix}$

We have standard projective resolutions:

M $\quad 0 \longrightarrow 2 \longrightarrow \begin{smallmatrix}3\\2\end{smallmatrix} \longrightarrow 3 \longrightarrow 0$

M' $\quad 0 \longrightarrow 2 \oplus 2 \longrightarrow \begin{smallmatrix}1\\2\end{smallmatrix} \oplus \begin{smallmatrix}3\\2\end{smallmatrix} \oplus 2 \longrightarrow \begin{smallmatrix}1&&3\\&2&\end{smallmatrix} \longrightarrow 0$

Observe that the res. of M' is <u>not</u> minimal. We eliminate $S(2)$ from each of the proj. modules.

M' $\quad 0 \longrightarrow 2 \longrightarrow \begin{smallmatrix}1\\2\end{smallmatrix} \oplus \begin{smallmatrix}3\\2\end{smallmatrix} \longrightarrow \begin{smallmatrix}1&&3\\&2&\end{smallmatrix} \longrightarrow 0$

<u>Defn.</u> Let $M \in \operatorname{rep} Q$. A <u>projective cover</u> of M is a proj. rep P together w/ surj. $g : P \twoheadrightarrow M$ s.t. whenever $g' : P' \twoheadrightarrow M$ (surj.) w/ P' proj, there exists surj. morphism $h : P' \twoheadrightarrow P$ s.t. the following commutes ($gh = g'$).

$$\begin{tikzcd}
 & P' \arrow[dl, dashed, two heads, "\exists h"'] \arrow[d, "g'"] & \\
P \arrow[r, two heads, "g"'] & M \arrow[r] \arrow[d] & 0 \\
 & 0 &
\end{tikzcd}$$

<u>Defn.</u> Let $M \in \operatorname{rep} Q$. An <u>injective envelope</u> of M is an inj. rep I together w/ inj. $f: M \hookrightarrow I$ s.t. whenever $f': M \hookrightarrow I'$ (inj) w/ I' inj, there exists inj. morphism $h: I \hookrightarrow I'$ s.t. the following diagram commutes ($hf = f'$).

$$\begin{tikzcd} & 0 \arrow[d] & \\ 0 \arrow[r] & M \arrow[r, hook, "f"] \arrow[d, hook, "f'"'] & I \arrow[dl, dashed, hook, "h"] \\ & I' & \end{tikzcd}$$

<u>Defn.</u> A proj. resolution is <u>minimal</u> if $f_0: P_0 \twoheadrightarrow M$, $f_i: P_i \twoheadrightarrow \ker f_{i-1}$ are projective covers.

$$\cdots \xrightarrow{f_3} P_2 \xrightarrow{f_2} P_1 \xrightarrow{f_1} P_0 \xrightarrow{f_0} M \longrightarrow 0$$

An inj. res. is <u>minimal</u> if $f_0: M \hookrightarrow I_0$, $f_i: \operatorname{coker} f_{i-1} \hookrightarrow I_i$ are injective envelopes

$$0 \longrightarrow M \xrightarrow{f_0} I_0 \xrightarrow{f_1} I_1 \xrightarrow{f_2} I_2 \xrightarrow{f_3} \cdots$$

<u>Prop.</u> Let $g: P \twoheadrightarrow M$ be a proj. cover of M & $g': P' \twoheadrightarrow M$ (surj) w/ P' proj. Then $P \cong P' \oplus \cdots \oplus P'$.

<u>Pf.</u> $\exists\, h: P' \twoheadrightarrow P \Rightarrow$ the following exact seq.

$$0 \longrightarrow \ker h \longrightarrow P' \xrightarrow{h} P \longrightarrow 0$$

P proj $\Rightarrow$ result follows from result previous in (2.1). $\square$

Prop. Let $g: P \twoheadrightarrow M$ & $g': P' \twoheadrightarrow M$ be proj. covers of M. Then $P \simeq P'$.

Pf. from above we get $P \simeq P' \oplus \cdots \oplus P'$ & $P' \simeq P \oplus \cdots \oplus P$

$\Longrightarrow P \simeq P'$ ☐

Defn. Let $A := \bigoplus_{i \in Q_0} P(i)$. A representation is called free if $F \in \operatorname{rep} Q$ is s.t. $F \simeq A \oplus \cdots \oplus A$.

Prop. A rep $M \in \operatorname{rep} Q$ is proj. iff $\exists F \in \operatorname{rep} Q$ s.t. F free & M is isomorphic to direct summand of F.

Pf. ($\Leftarrow$) Every direct sum of F is direct sum of $P(i)$s $\Longrightarrow$ its proj.

($\Longrightarrow$) Sps M proj. & $\underline{\dim} M = (d_i)_{i \in Q_0}$, standard proj. res gives surj. morphism $g: \bigoplus d_i P(i) \twoheadrightarrow M$. So,

$$0 \longrightarrow \ker g \longrightarrow \bigoplus d_i P(i) \xrightarrow{g} M \longrightarrow 0$$

M proj $\Longrightarrow$ seq. splits & $M \simeq \bigoplus d_i P(i)$. ☐

Cor. Any proj. rep $P \in \operatorname{rep} Q$ is a direct sum of $P(i)$s

$$P \simeq P(i_1) \oplus \cdots \oplus P(i_t)$$

w/ $i_1, \ldots, i_t$ not necessarily distinct.

<u>Defn.</u> Let $P(i) = (P(i)_j, \varphi_\alpha)$ be the proj. rep at i. The <u>radical</u> of $P(i)$ is the rep. $\mathrm{rad}(P(i)) = (R_j, \varphi'_\alpha)$ defined as

$$R_i = 0 \qquad R_j = P(i)_j \text{ if } i \neq j \quad \text{and} \quad \varphi'_\alpha = \begin{cases} 0 & \text{if } s(\alpha) = i \\ \varphi_\alpha & \text{otherwise} \end{cases}$$

<u>Lem.</u> Any proper subrep of $P(i)$ is contained in rad $P(i)$.

<u>Pf.</u> Spse $f: M \hookrightarrow P(i)$ w/ $M_i \neq 0$. we show f is an isomorphism.
$P(i)_i \simeq \Bbbk \Rightarrow M_i \simeq \Bbbk$, $\exists\, m_i \in M_i$ s.t. $f_i(m_i) = e_i$. Let j be a vertex & $c = (i|\cdots|j)$. Then,

$$c = \varphi_c(e_i) = \varphi_c(f_i(m_i)) = f_j(\psi^s_c(m_i)) \in \mathrm{im} f$$

Arbitrary elem c of basis of $P(i)_j \in \mathrm{im}\, f_j \Rightarrow f$ surj. M not proper subrep of $P(i)$ $\square$

<u>Lem.</u> If $P(i)$ is simple then rad $P(i) = 0$. If $P(i)$ is not simple then rad $P(i)$ is projective.

<u>Pf.</u> Will show $\mathrm{rad}\, P(i) \simeq P = \bigoplus_{\alpha : s(\alpha) = i} P(t(\alpha))$.
if $i \neq j$ then $(\mathrm{rad}\, P(i))_j = P(i)_j$ w/ basis the set of paths from i to j.

Define f.

$$f = (f_j)_{j \in Q_0} : \mathrm{rad}\, P(i) \longrightarrow P \text{ s.t. } f_j : (i|\alpha, \cdots|j) \longmapsto (t(\alpha)|\cdots|j).$$

Each f_j sends a basis of $(\mathrm{rad}\, P(i))_j$ to a basis of $P_j \Rightarrow f$ iso. $\square$

Thm. Subreps of proj. reps in rep$Q$ are projective

(rep$Q$ is hereditary, subrep inherets projectivity)

Pf. Sps P proj. rep, we prove using induction on dimension of P (Here d is dimension of P defined as $d = \sum_{i \in Q_0} d_i$, for $d_i \in \underline{\dim} P$).

If $d=1$ then P is simple $\Rightarrow$ nothing to prove.

Sps $d>1$. Let M be a subrep of P & $\upsilon: M \to P$ the inclusion morphism
$P \simeq P(i_1) \oplus \cdots \oplus P(i_t)$ for some $i_j \in Q_0$ $\Rightarrow$ υ is of the form
$\upsilon = \begin{bmatrix} \upsilon_1 \\ \vdots \\ \upsilon_t \end{bmatrix}$ w/ $\operatorname{im} \upsilon_j \subset P(i_j)$. Follows that $M \simeq \operatorname{im} \upsilon_1 \oplus \cdots \oplus \operatorname{im} \upsilon_t$
thus suffices to show that $\operatorname{im} \upsilon_j$ is proj. for each j. This is true obviously
when $\operatorname{im} \upsilon_j = P(i_j)$, sps $\operatorname{im} \upsilon_j$ is proper subrep of $P(i_j)$
$\Rightarrow$ $\operatorname{im} \upsilon_j$ is subrep of rad $P(i_j)$. rad $P(i_j)$ is proj. & its dimension is strictly smaller than d, by induction we see that $\operatorname{im} \upsilon_j$ is proj. $\square$

Cor. Let $f: M \to P$, $f \neq 0$, M indecomp., P proj. Then M is proj. & f injective.

Pf. $\operatorname{im} f$ is subrep of P $\Rightarrow$ $\operatorname{im} f$ proj. so $0 \to \ker f \to M \to \operatorname{im} f \to 0$
splits. Moreover, $\operatorname{im} f \simeq M \oplus \cdots \oplus M$ (direct summand of M)
But M indecomp. so $M \simeq \operatorname{im} f$ is projective & $\ker f = 0$. $\square$

2.3 Auslander-Reiten Translation

<u>Defn.</u> Let $\mathcal{C}, \mathcal{D}$ be two categories w/ $F_1, F_2 : \mathcal{C} \to \mathcal{D}$ as two functors. We say F_1 & F_2 are <u>functorially isomorphic</u> ($F_1 \simeq F_2$) if for all $M \in \mathcal{C}$, $\exists\ \eta_M : F_1(M) \to F_2(M) \in \mathcal{D}$ s.t. for $f : M \to N \in \mathcal{C}$ the following commutes

$$\begin{tikzcd} F_1(M) \arrow[r, "F_1(f)"] \arrow[d, "\eta_M"'] & F_1(N) \arrow[d, "\eta_N"] \\ F_2(M) \arrow[r, "F_2(f)"] & F_2(N) \end{tikzcd}$$

A covariant functor $F : \mathcal{C} \to \mathcal{D}$ is called an <u>equivalence of categories</u> if there exists a functor $G : \mathcal{D} \to \mathcal{C}$ s.t. $G \circ F \simeq 1_{\mathcal{C}}$ & $F \circ G \simeq 1_{\mathcal{D}}$

The functor G is called a <u>quasi-inverse</u> functor for F. Here F is called a <u>Duality</u>.

Q^{op} is the quiver associated w/ Q where the α's are reversed

i.e.

$$Q^{op} = (Q_0, Q_1^{op}) \quad \text{w/ } Q_1^{op} := \left\{ \alpha^{op} \in Q_1 \;\middle|\; \begin{array}{l} s(\alpha^{op}) = t(\alpha) \\ t(\alpha^{op}) = s(\alpha) \end{array} \right\}$$

The Duality $D = \mathrm{Hom}_{\Bbbk}(-,\Bbbk) : \mathrm{rep}\,Q \longrightarrow \mathrm{rep}\,Q^{op}$

is the contravariant functor defined as follows:

for objects — $M \in \mathrm{rep}\,Q$, $M = (M_i, \varphi_\alpha)$

$$DM = (DM_i, D\varphi_\alpha^{op})_{i \in Q_0,\ \alpha \in Q_1}$$

w/ DM_i is the *dual vector space* of $M_i \Rightarrow DM_i = \mathrm{Hom}_{\Bbbk}(M_i, \Bbbk)$ space of linear functionals (maps) $M_i \to \Bbbk$. If α is an arrow in Q then $D\varphi_\alpha^{op}$ is the *pullback* of φ_α. Thus

$$D\varphi_\alpha^{op} : DM_{t(\alpha)} \longrightarrow DM_{s(\alpha)} \quad \text{s.t.} \quad u \longmapsto u \circ \varphi_\alpha.$$

for morphisms — $f : M \to N \in \mathrm{rep}\,Q$ we have $Df : DN \to DM \in \mathrm{rep}\,Q^{op}$ defined as $Df(u) = u \circ f$.

$$\begin{tikzcd}
M \arrow[r, "f"] \arrow[dr, "u \circ f"'] & N \arrow[d, "u"] \\
 & \Bbbk
\end{tikzcd}$$

Composing the duality of Q w/ duality of Q^{op} we get the identity functor $1_{\mathrm{rep}\,Q}$. The quasi-inverse of D_Q is $D_{Q^{op}}$.

Prop. $D(P_Q(i)) = I_{Q^{op}}(i)$ $\forall i \in Q_0$, The duality restricts to a duality

$$\mathrm{proj}\,Q \longrightarrow \mathrm{inj}\,Q^{op}.$$

Ex. $Q: 1 \longrightarrow 2 \longleftarrow 3$. The indecomp. reps in $\text{proj}\,Q \subset \text{rep}\,Q$.

$P(1) = [1,1,0]$, $P(2) = [0,1,0]$, $P(3) = [0,1,1]$

Then $Q^{op}: 1 \longleftarrow 2 \longrightarrow 3$. The indecomp. reps in $\text{inj}\,Q^{op} \subset \text{rep}\,Q^{op}$

$I(1) = [1,1,0]$, $I(2) = [0,1,0]$, $I(3) = [0,1,1]$

Or in another notation: $\begin{smallmatrix}1\\2\end{smallmatrix} \quad 2 \quad \begin{smallmatrix}3\\2\end{smallmatrix} \longrightarrow \begin{smallmatrix}2\\1\end{smallmatrix} \quad 2 \quad \begin{smallmatrix}2\\3\end{smallmatrix}$.

Nakayama Functor

Let $A := \bigoplus_{j \in Q_0} P(j)$ be the free rep. of indecomposable proj. reps

Recall contravariant functor $\text{Hom}(-, A)$. We give $\text{Hom}(X, A)$ the structure of a rep $(M_i, \varphi_{\alpha^{op}}) \in \text{rep}\,Q^{op}$ as follows.

$$M_i := \text{Hom}(X, P(i))$$

$$\varphi_{\alpha^{op}} := \text{Hom}(X, P(j)) \longrightarrow \text{Hom}(X, P(i)) \text{ s.t. } \varphi_{\alpha^{op}}(f) = \alpha \circ f$$

for arrows $\alpha: i \longrightarrow j \in Q$. We get the diagram:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "\varphi_{\alpha^{op}}(f)"'] & P(j) \arrow[d, "\alpha"] \\ & P(i) \end{tikzcd}$$

Since $\alpha: i \longrightarrow j$ gives a morphism $P(j) \longrightarrow P(i)$

To see $\text{Hom}(-, A)$ is a functor from $\text{rep}\,Q$ to $\text{rep}\,Q^{op}$ NTS image of $\text{Hom}(-, A)$ of $g: X \longrightarrow X' \in \text{rep}\,Q$ is a morphism of reps $\in \text{rep}\,Q^{op}$.

For every $i \xrightarrow{\alpha} j \in Q$ the following commutes.

$$\begin{tikzcd} \text{Hom}(X', P(j)) \arrow[r, "\varphi'_{\alpha^{op}}"] \arrow[d, "g^* = \text{Hom}(g, P(j))"'] & \text{Hom}(X', P(i)) \arrow[d, "g^* = \text{Hom}(g, P(i))"] \\ \text{Hom}(X, P(j)) \arrow[r, "\varphi_{\alpha^{op}}"'] & \text{Hom}(X, P(i)) \end{tikzcd}$$

Let $f \in \mathrm{Hom}(X', P(j))$ then $g^* \varphi'_{\alpha^{op}}(f) = g^*(\alpha \circ f) = (\alpha \circ f) \circ g$

whereas $\varphi_{\alpha^{op}} \circ g^*(f) = \varphi_{\alpha^{op}}(f \circ g) = \alpha \circ (f \circ g) = (\alpha \circ f) \circ g$ we have this,

Prop. $\mathrm{Hom}(-, A)$ is a functor from $\mathrm{rep}\,Q$ to $\mathrm{rep}\,Q^{op}$.

We now compose D & $\mathrm{Hom}(-, A)$ we get the following covariant functor.

Defn. The functor $\nu = D\,\mathrm{Hom}(-, A) : \mathrm{rep}\,Q \longrightarrow \mathrm{rep}\,Q$ is called the nakayama functor

$$\begin{tikzcd} \mathrm{rep}\,Q \arrow[r, "\mathrm{Hom}(-,A)"] \arrow[rr, bend right, "\nu"'] & \mathrm{rep}\,Q^{op} \arrow[r, "D"] & \mathrm{rep}\,Q \end{tikzcd}$$

Let $DA^{op} = \bigoplus_{i \in Q_0} I_Q(i)$ where A^{op} is the sum of all indecomposable Q^{op}-reps.

Prop. The restriction of ν to $\mathrm{proj}\,Q$ is an equivalence of categories $\mathrm{proj}\,Q \longrightarrow \mathrm{inj}\,Q$, w/ quasi-inverse given as

$$\nu^{-1} = \mathrm{Hom}(DA^{op}, -) : \mathrm{inj}\,Q \longrightarrow \mathrm{proj}\,Q.$$

Moreover, for all $i \in Q_0$ $\nu P(i) = I(i)$. If $c = (i|\cdots|j)$ & $f_c : P(j) \to P(i) \in \mathrm{Hom}(P(j), P(i))$ corresponds to c then

$$\nu f_c : I(j) \longrightarrow I(i)$$

is the morphism given by the cancelation of the path c.

Pf.

Equivalence follows since

$$\nu = D \circ \mathrm{Hom}(-, A) \implies \nu^{-1} = \mathrm{Hom}(-, A^{op}) \circ D$$

quasi-inverses $\downarrow$ $\quad$ $\downarrow$

$\hookrightarrow D \quad \mathrm{Hom}(-, A^{op})$

Then $\mathrm{Hom}_{Q^{op}}(DX, DY) \simeq \mathrm{Hom}_Q(Y, X)$; $X, Y \in \mathrm{rep}\,Q$

$\implies \mathrm{Hom}_{Q^{op}}(DX, A^{op}) \simeq \mathrm{Hom}_Q(DA^{op}, X)$

$\implies \nu P_Q(i) = D\,\mathrm{Hom}(P_Q(i), A) = D(P_{Q^{op}}(i)) = I_Q(i)$

This completes the first statement.

Let $c = (i | \cdots | j)$, $f_c : P_Q(j) \longrightarrow P_Q(i)$, $f(x) = cx$

Let f_c^* be the image of f_c under functor $\mathrm{Hom}(-, A)$

$f_c^* : \mathrm{Hom}(P_Q(i), A) \longrightarrow \mathrm{Hom}(P_Q(j), A)$ maps a morphism g to the pullback $g \circ f_c^*$. Then since $\mathrm{Hom}(P_Q(x), A) \simeq P_{Q^{op}}(x)$

$\implies f_c^* : P_{Q^{op}}(i) \longrightarrow P_{Q^{op}}(j) \implies f(y) = c^{op} y$

Then $\nu f_c = D f_c^*$ is the map sending $D(c^{op} y)$ to $D(y)$

So $D(c^{op} y) = D(y) c \implies \nu f_c : I(j) \longrightarrow I(i)$. $\square$

Let $Q: 1 \to 2 \to 3$ then,

$$\Gamma_Q: \quad \begin{tikzcd}[column sep=small] & & P(1)=I(3) \arrow[dr] & & \\ & P(2) \arrow[ur] \arrow[dr] & & I(2) \arrow[dr] & \\ P(3)=S(3) \arrow[ur] & & S(2) \arrow[ur] & & I(1)=S(1) \end{tikzcd}$$

Where Nakayama sends proj Q to inj Q as below:

$$\begin{tikzcd}[column sep=small] & & P(1)=\nu P(3) \arrow[dr] & & \\ & P(2) \arrow[ur] & & \nu P(2) \arrow[dr] & \\ P(3) \arrow[ur] & & & & \nu P(1) \end{tikzcd}$$

<u>Defn.</u> Let $\mathcal{C}$ & $\mathcal{D}$ be abelian categories. A functor $F: \mathcal{C} \to \mathcal{D}$ is called <u>exact</u> if it maps exact seq. to exact seq. Let $F: \mathcal{C} \to \mathcal{D}$ be covariant, and $G: \mathcal{C} \to \mathcal{D}$ be contravariant.

<u>left exact</u>

$$0 \to L \xrightarrow{f} M \xrightarrow{g} N \iff 0 \to F(L) \xrightarrow{F(f)} F(M) \xrightarrow{F(g)} F(N)$$

$$L \xrightarrow{f} M \xrightarrow{g} N \to 0 \iff 0 \to G(N) \xrightarrow{G(g)} G(M) \xrightarrow{G(f)} G(L)$$

<u>right exact</u>

$$L \xrightarrow{f} M \xrightarrow{g} N \to 0 \iff F(L) \xrightarrow{F(f)} F(M) \xrightarrow{F(g)} F(N) \to 0$$

$$0 \to L \xrightarrow{f} M \xrightarrow{g} N \iff G(N) \xrightarrow{G(g)} G(M) \xrightarrow{G(f)} G(L) \to 0$$

We know $\mathrm{Hom}(X,-)$ & $\mathrm{Hom}(-,X)$ are left exact.
Similarly, D is exact contravariant functor. Then, since
$\nu := D \circ \mathrm{Hom}(-,A) \implies$

<u>Prop.</u> The Nakayama functor ν is right exact.

Continuing on from previous ex. we have a short exact seq.

$$0 \longrightarrow P(3) \xrightarrow{f} P(1) \xrightarrow{g} I(2) \longrightarrow 0$$

Then, applying ν gives us the exact seq.

$$P(1) \xrightarrow{\nu f} I(1) \xrightarrow{\nu g} 0 \longrightarrow 0$$

Since νf is not inj. this shows that ν is not exact.

<u>The Auslander-Reiten Translations τ, τ^{-1}</u>

Let Q be a quiver w/o orientated cycles, and M an indecomp. rep of Q.

<u>Defn.</u> Let $0 \longrightarrow P_1 \xrightarrow{p_1} P_0 \xrightarrow{p_0} M \longrightarrow 0$ be a minimal proj. res. Applying Nakayama functor we get the exact seq.

$$0 \longrightarrow \tau M \longrightarrow \nu P_1 \xrightarrow{\nu p_1} \nu P_0 \xrightarrow{\nu p_0} \nu M \longrightarrow 0$$

where $\tau M = \ker \nu p_1$ is the Auslander-Reiten <u>translate</u> of M.
And τ the <u>Auslander-Reiten translation.</u>

Let $0 \longrightarrow M \xrightarrow{i_0} I_0 \xrightarrow{i_1} I_1 \longrightarrow 0$ be a minimal inj. res. Applying inverse Nakayama functor we get the exact seq.

$$0 \longrightarrow \nu^{-1}M \xrightarrow{\nu^{-1}i_0} \nu^{-1}I_0 \xrightarrow{\nu^{-1}i_1} \nu^{-1}I_1 \longrightarrow \tau^{-1}M \longrightarrow 0$$

where $\tau^{-1}M = \operatorname{coker} \nu^{-1}i_1$ is the Auslander-Reiten <u>translate</u> of M. And τ^{-1} the <u>Auslander-Reiten translation</u>.

<u>Ex.</u> Consider the minimal proj. res of $\begin{smallmatrix}1\\2\end{smallmatrix}$ as in example $Q := 1 \to 2 \to 3$.

$$0 \longrightarrow 3 \xrightarrow{f} \begin{smallmatrix}1\\2\\3\end{smallmatrix} \xrightarrow{g} \begin{smallmatrix}1\\2\end{smallmatrix} \longrightarrow 0$$

Applying ν we get

$$\begin{smallmatrix}1\\2\\3\end{smallmatrix} \xrightarrow{\nu f} 1 \xrightarrow{\nu g} 0 \longrightarrow 0$$

remains to compute $\tau \begin{smallmatrix}1\\2\end{smallmatrix}$

$$\tau \begin{smallmatrix}1\\2\end{smallmatrix} = \ker \nu f = \begin{smallmatrix}2\\3\end{smallmatrix}$$

2.4 Extensions & Ext

Sps Q is finite & no orientated cycles. Let $M \in \mathrm{rep}\,Q$.

$$0 \longrightarrow P_1 \xrightarrow{f} P_0 \xrightarrow{g} M \longrightarrow 0 \qquad (1)$$

is a proj. res. Let $N \in \mathrm{rep}\,Q$. Apply $\mathrm{Hom}(-,N)$ to (1) to get.

$$\cancel{0 \longrightarrow \mathrm{Hom}(M,N) \longrightarrow \mathrm{Hom}(P_0,N) \longrightarrow \mathrm{Hom}(P_1,N) \longrightarrow 0}$$

$$0 \longrightarrow \mathrm{Hom}(M,N) \xrightarrow{g^*} \mathrm{Hom}(P_0,N) \xrightarrow{f^*} \mathrm{Hom}(P_1,N) \longrightarrow \mathrm{Ext}^1(M,N) \longrightarrow 0$$

where $\mathrm{Ext}^1(M,N) = \mathrm{coker} f^*$ is called the <u>first group of extensions</u> of M, N.

For arbitrary categories proj. (inj.) res. might not stop after 2 steps.
Applying $\mathrm{Hom}(-,N)$ gives a <u>co-chain complex</u>

$$0 \longrightarrow \mathrm{Hom}(M,N) \xrightarrow{f_0^*} \mathrm{Hom}(P_0,N) \xrightarrow{f_1^*} \cdots \xrightarrow{f_n^*} \mathrm{Hom}(P_n,N) \longrightarrow$$

where $f_i^* f_{i-1}^* = 0 \ \forall i$. For $i \geq 1$ the i^{th} extension group is

$$\mathrm{Ext}^i(M,N) = \ker f_{i+1}^* / \mathrm{im}\, f_i^*$$

In $\mathrm{rep}\,Q$ all Ext^i-groups vanish for $i \geq 2$

<u>Defn.</u> An <u>extension</u> ξ of M by N is a short exact seq.

$$0 \to N \xrightarrow{f} E \xrightarrow{g} M \to 0$$

Two extensions ξ ξ' are <u>equivalent</u> if the following commutes:

$$\begin{tikzcd}
\xi : 0 \arrow[r] & N \arrow[r, "f"] \arrow[d, "="] & E \arrow[r, "g"] \arrow[d, "\simeq"] & M \arrow[r] \arrow[d, "="] & 0 \\
\xi' : 0 \arrow[r] & N \arrow[r, "f'"] & E' \arrow[r] & M \arrow[r] & 0
\end{tikzcd}$$

<u>Ex.</u> Let Q be $1 \overset{\alpha}{\underset{\beta}{\rightrightarrows}} 2$ with $N = S(2)$ & $M = S(1)$.

w/ $E = k \overset{1}{\underset{0}{\rightrightarrows}} k$ & $E' = k \overset{0}{\underset{1}{\rightrightarrows}} k$. Then the extensions are

<u>not</u> equivalent:

$$\xi : 0 \to S(2) \xrightarrow{f} E \xrightarrow{g} S(1) \to 0$$

$$\xi' : 0 \to S(2) \xrightarrow{f'} E' \xrightarrow{g'} S(1) \to 0$$

since $E \not\simeq E'$.

An extension is <u>split</u> if $E \simeq N \oplus M$.

Define $\xi + \xi'$ as follows: Let $E'' := \{(x, x') \in E \times E' \mid g(x) = g'(x')\}$ be the pull back of g & g'. Let F be the quotient of E'' where $\{(f(n), -f'(n)) \in E \oplus E' \mid n \in N\}$. Then $\xi + \xi'$ is

$$0 \to N \to F \to M \to 0.$$

$\mathcal{E}(M,N)$ of extensions of M by N is the set of equivalence classes of extensions. This forms an <u>abelian group</u> w/ operation above. We have an isomorphism of groups

$$\mathcal{E}(M,N) \longrightarrow \mathrm{Ext}^1(M,N)$$

Let $\xi : 0 \to N \xrightarrow{u'} E \xrightarrow{v'} M \to 0 \in \mathcal{E}(M,N)$ and $0 \to P_1 \xrightarrow{f} P_0 \xrightarrow{g} M \to 0$ a proj. res. Then P_0 proj. $\Rightarrow$ $\exists f' \in \mathrm{Hom}(P_0, E)$ s.t. $g = v'f'$. Then ξ exact $\Rightarrow \exists\, u \in \mathrm{Hom}(P_1, N)$ s.t.

$$\begin{tikzcd}
 & P_1 \arrow[r, "f"] \arrow[d, "u"] & P_0 \arrow[r, "g"] \arrow[d, "f'"] & M \arrow[r] \arrow[d, "="] & 0 \\
\xi : 0 \arrow[r] & N \arrow[r, "u"] & E \arrow[r, "v'"] & M \arrow[r] & 0
\end{tikzcd}$$

commutes. Then, $\mathrm{Ext}^1(M,N) = \mathrm{coker}\, f^* = \mathrm{Hom}(P_1,N)/\mathrm{im}\, f^*$.

That is, the isomorphism $\mathcal{E}(M,N) \longrightarrow \mathrm{Ext}^1(M,N)$ sends the class ξ to the class of u.

Continued previously done example on next page.

Ex. Let $Q: 1 \underset{\beta}{\overset{\alpha}{\rightrightarrows}} 2$

$N = S(2)$, $M = S(1)$, $E = k \underset{0}{\overset{1}{\rightrightarrows}} k$, $E' = k \underset{1}{\overset{0}{\rightrightarrows}} k$

Using $E = (E_i, \varphi_\alpha)$ & $E' = (E'_i, \varphi'_\alpha)$ then

$$E_1 \simeq E_2 \simeq k \quad \varphi_\alpha = 1 \quad \varphi_\beta = 0$$

$$E'_1 \simeq E'_2 \simeq k \quad \varphi'_\alpha = 0 \quad \varphi_\beta = 1$$

Need to compute pull back E'': $E'' = \{((e_1, e_2), (e'_1, e'_2)) \in E \times E'\}$ s.t. $g(e_1, e_2) = g'(e'_1, e'_2)$. Since g & g' projs. on first component we have

$$E'' = \{((e_1, e_2), (e_1, e'_2)) \in E \times E'\}$$

Computing φ''_α & φ''_β we see

$$\varphi''_\alpha((e_1, e_2), (e_1, e'_2)) = (\varphi_\alpha(e_1, e_2), \varphi'_\alpha(e_1, e'_2)) = ((0, e_1), (0, 0))$$

$$\varphi''_\beta((e_1, e_2), (e_1, e'_2)) = (\varphi_\beta(e_1, e_2), \varphi'_\beta(e_1, e'_2)) = ((0, 0), (0, e_1))$$

i.e. $E'' = k \underset{\begin{bmatrix}0\\1\end{bmatrix}}{\overset{\begin{bmatrix}1\\0\end{bmatrix}}{\rightrightarrows}} k^2$. Computing $F = E'' / \{((0, n), (0, -n)) \mid n \in k\}$

Then $F_1 \simeq F_2 \simeq k$. Ultimately, $F \simeq k \underset{-1}{\overset{1}{\rightrightarrows}} k$. All together we see $\xi + \xi'$ is the short exact seq.

$$0 \longrightarrow S(2) \xrightarrow{f''} F \xrightarrow{g''} S(1) \longrightarrow 0$$

f'' is the incl. of 2^{nd} comp. & g'' is the proj on the 1^{st} comp.

Ch. 3 Examples of Auslander-Reiten Quivers

3.1 Auslander-Reiten Quivers of Type $\mathbb{A}_n$

The goal is to construct Γ_Q when Q is of underlying Dynkin type $\mathbb{A}_n$.

That is of the form:

$$1 \text{ --- } 2 \longrightarrow 3 \text{ --- } \cdots \text{ --- } (n-1) \text{ --- } n$$

The Knitting Algorithm

1) Compute all indec. projective reps $P(1), P(2), \ldots$
2) Draw an arrow $P(i) \longrightarrow P(j)$ whenever $\exists\ j \longrightarrow i \in Q_1$ s.t. each $P(i)$ sits at a different level.
3) (Knitting) Complete each mesh of the forms

$$\begin{tikzcd}[column sep=small, row sep=small] & M_1 \\ L \arrow[ur] \arrow[dr] \\ & M_2 \end{tikzcd} \rightsquigarrow \begin{tikzcd}[column sep=small, row sep=small] & M_1 \arrow[dr] \\ L \arrow[ur] \arrow[dr] & & \tau^{-1}L \\ & M_2 \arrow[ur] \end{tikzcd}, \quad \begin{tikzcd}[column sep=small, row sep=small] & M \\ L \arrow[ur] \end{tikzcd} \rightsquigarrow \begin{tikzcd}[column sep=small, row sep=small] & M \arrow[dr] \\ L \arrow[ur] & & \tau^{-1}L \end{tikzcd}, \quad \begin{tikzcd}[column sep=small, row sep=small] L \arrow[dr] \\ & M \end{tikzcd} \rightsquigarrow \begin{tikzcd}[column sep=small, row sep=small] L \arrow[dr] & & \tau^{-1}L \\ & M \arrow[ur] \end{tikzcd}$$

s.t. $\underline{\dim}\, L + \underline{\dim}\, \tau^{-1}L = \sum_{i=1}^{2} \underline{\dim}\, M_i$.

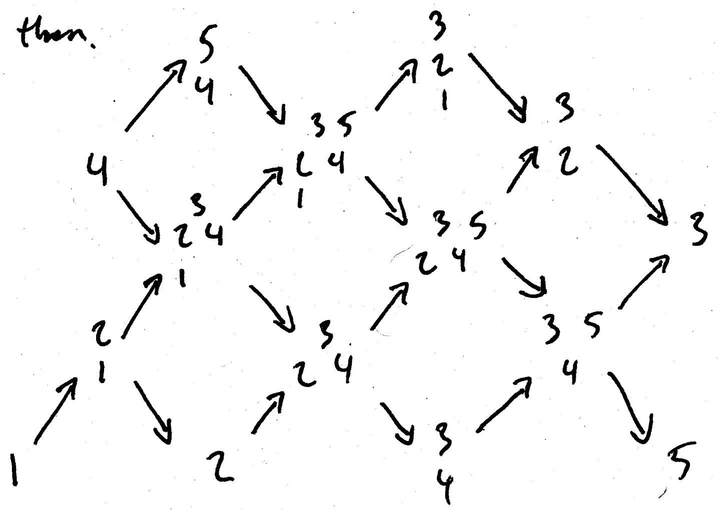
4) repeat 3) until negative integers are found in dim vector.

<u>Ex.</u>

Let $Q := 1 \longleftarrow 2 \longleftarrow 3 \longrightarrow 4 \longleftarrow 5.$

It follows that $P(1) = 1$ $\quad P(2) = \begin{smallmatrix} 2 \\ 1 \end{smallmatrix}$ $\quad P(3) = \begin{smallmatrix} & 3 & \\ 2 & & 4 \\ 1 & & \end{smallmatrix}$ $\quad P(4) = 4$ $\quad P(5) = \begin{smallmatrix} 5 \\ 4 \end{smallmatrix}$

& Γ_Q is then.



<u>τ-orbits:</u> τ is the Auslander-Reiten translation, in Γ_Q τ sends the rightmost point of a mesh to the leftmost point of the same mesh.

We call the set of repeatedly applying τ or τ^{-1} to an indec. rep. the τ-orbit of that indec. rep. $\Longrightarrow$ τ-orbits for A_n quivers is the reps. on the same level of Γ_Q.

Each τ-orbit consists of exactly <u>one</u> proj. rep, thus we can compute the whole of Γ_Q by using τ-orbits on $P(i)$. There are several methods of computing τ-orbits.

- Auslander-Reiten Translation
- Coxeter Functor
- Polygon Diagonals.

Method 1) AR-translation

Let M be indec. rep & not injective. We want to find the translation $\tau^{-1}M$ (to the right) of M. Start w/ inj. resolution

$$0 \longrightarrow M \longrightarrow I_0 \xrightarrow{\ g\ } I_1 \longrightarrow 0$$

Apply inverse nakayama functor ν^{-1} (Mapping indec. $\underline{I}(-,j)$ to indec. $P(-,j)$)

$$0 \longrightarrow \nu^{-1}I_0 \xrightarrow{\ \nu^{-1}(g)\ } \nu^{-1}I_1 \longrightarrow \tau^{-1}M \longrightarrow 0.$$

Consider the module $M = 4$ as in the example. Then,

$$\begin{tikzcd}
0 \arrow[r] & 4 \arrow[r] & \begin{smallmatrix}3&&5\\&4&\end{smallmatrix} \arrow[r] \arrow[d, "\nu^{-1}"] & 3 \oplus 5 \arrow[r] \arrow[d, "\nu^{-1}"] & 0 & \\
& 0 \arrow[r] & 4 \arrow[r] & \begin{smallmatrix}&3\\2&4\\1&\end{smallmatrix} \oplus \begin{smallmatrix}5\\4\end{smallmatrix} \arrow[r] & \begin{smallmatrix}&3&5\\2&4&\\1&&\end{smallmatrix} \arrow[r] & 0
\end{tikzcd}$$

So $\tau^{-1}M = \begin{smallmatrix}&3&5\\2&4&\\1&&\end{smallmatrix}$. This can be done for all reps until Γ_Q is complete.

Method 2 Coxeter Functor

Choose a seq. $(i_1, i_2, \dots, i_n)$ w/ $i_j = i_\ell$ if $i \neq \ell$ as follows:

$i_1 :=$ a sink of Q

$i_2 := i_2$ is a sink of $s_i Q = Q$ reversing arrows incident to vertex i_1.

$\vdots$

$i_t :=$ a sink of $s_{i_{t-1}} \cdots s_{i_2} s_{i_1} Q$

As w/ previous ex. $(1, 4, 2, 3, 5)$ is a seq.

Define reflections $s_i : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ s.t. $s_i : x \longmapsto x - 2B(x, e_i) e_i$

where e_i a basis of $\mathbb{R}^n$ & B is the symmetric bilinear form:

$$B(e_i, e_j) = \begin{cases} 1 & i = j \\ -1/2 & i \text{ adj to } j \in Q \\ 0 & \text{otherwise.} \end{cases}$$

<u>Defn.</u> The <u>Coxeter elem.</u> $c = s_{i_1} s_{i_2} \cdots s_{i_n}$ is a product of reflections using the seq. of vertices above.

$\hookrightarrow (1, 4, 2, 3, 5) \overset{\text{Def}}{\Longrightarrow} c = s_1 s_4 s_2 s_3 s_5$

This then can be used to compute the $\underline{\dim}(\tau^{-1} M)$ from $\underline{\dim} M$

Then $c\left(\sum_i d_i e_i\right) = \sum_i d_i' e_i$ & $\underline{\dim}(\tau^{-1} M) = (d_1', \dots, d_n')$

Compute $\tau^{-1}M$, $\underline{\dim}M = (0,0,0,1,0)$ then

$$\underline{\dim}(\tau^{-1}M) = s_1 s_4 s_2 s_3 s_5 (e_4)$$
$$= s_1 s_4 s_2 s_3 (e_4 + e_5)$$
$$= s_1 s_4 s_2 (e_3 + e_4 + e_5)$$
$$= s_1 s_4 (e_2 + e_3 + e_4 + e_5) = e_1 + e_2 + e_3 + e_4 + e_5.$$

Method 3 Diagonals of a Polygon ($n+3$ vertices)

Start w/ regular polygon w/ $n+3$ vertices. A diagonal in the polygon is a straight line segment of polygon going through the interior.

A triangulation of a polygon is a maximal set of non-crossing diagonals. Such a triangulation of polygon cuts it into triangles.

Associate a triangulation T_Q to Q (A_n type quiver). Let 1 be a vertex in the quiver w/ one neighbour. Draw a diagonal that cuts off a triangle Δ_0 and label the diagonal 1. If $1 \leftarrow 2$ is an arrow in Q then draw the unique diagonal 2 s.t. $1, 2$, and one bounding form triangle s.t. diagonal 2 is clockwise of diagonal 1.

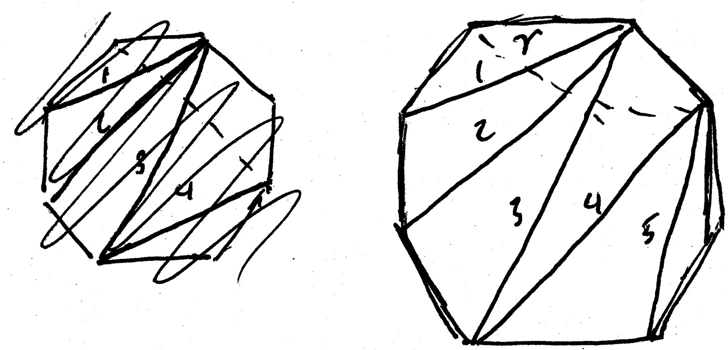
If $1 \rightarrow 2$ is an arrow draw diagonal 2 s.t. diagonal 2 is counterclockwise of diagonal 1.

Continue this up to diagonal n.

As w/ the previous example

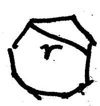
$$1 \leftarrow 2 \leftarrow 3 \rightarrow 4 \leftarrow 5$$

gives us.

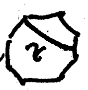
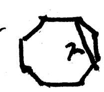


Any γ is a diagonal we associate $M_\gamma = (M_i, \varphi_\alpha)$ of Q by

$$M_i = \begin{cases} k & \text{if } \gamma \text{ crosses diagonal } i \\ 0 & \text{otherwise} \end{cases}$$

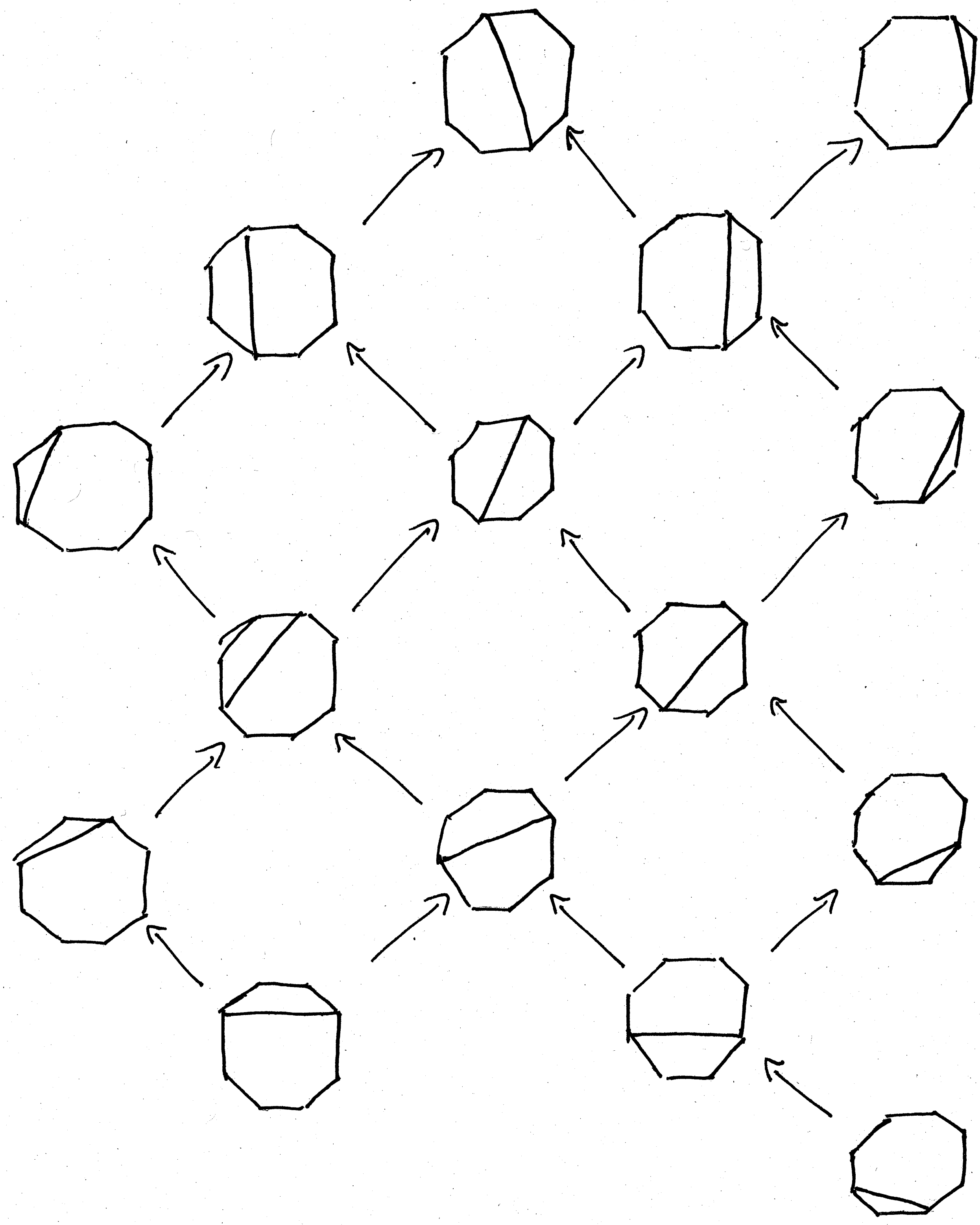
And $\varphi_\alpha = 1$ whenever $M_{s(\alpha)} = M_{t(\alpha)} = k$ and $\varphi_\alpha = 0$ otherwise. As above we have  crosses diagonals 1, 2, 3 corresponds to

$$M_\gamma = \mathbb{k} \xleftarrow{1} \mathbb{k} \xleftarrow{1} \mathbb{k} \xrightarrow{0} 0 \leftarrow 0.$$

Then Auslander-Reiten translation is given by elementary clockwise rotation. So $\tau($$)$ is the diagonal that cuts through diagonals 4 & 5 

$P(i)$ is given by τ^{-1} of the diagonal i. & $I(i)$ is given by τ of the diagonal i. In the example $P(1)$ is the diagonal cutting through diagonal 1 only & $I(i)$ is the diagonal γ.

The complete AR-quiver via polygons is on next page.

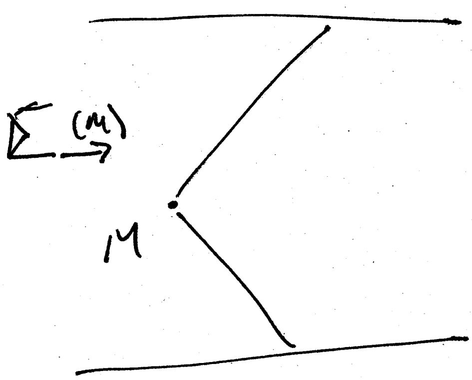


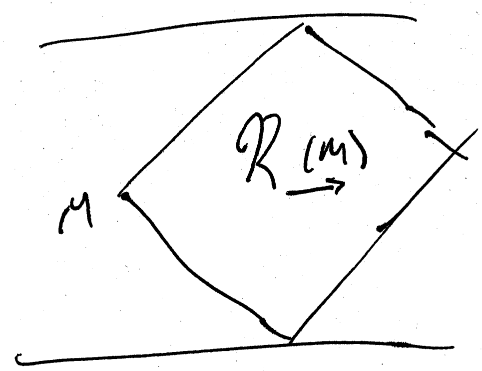
Computing Dimensions of Hom & Ext, & Short Exact Seqs

given $M, N \in \text{rep}\, Q$ we can easily compute dim. $\text{Hom}(M,N)$ from Γ_Q if M, N lie in the same component.

A path $M_0 \to M_1 \to \cdots \to M_s \in \Gamma_Q$ is called a <u>sectional path</u> if $\tau M_{i+1} \neq M_{i-1}$ for $1 \le i \le s-1$.

Let $\Sigma_{\rightarrow}(M)$ denote the set of all indecomposable reps. reached from M by a sectional path. Let $\Sigma_{\leftarrow}(M)$ be the set of all indec. reps. from which M may be reached by a sectional path.





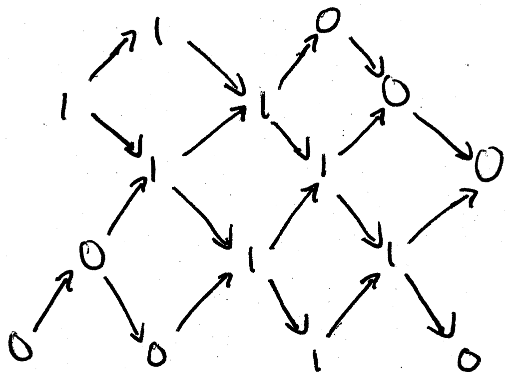
Let $\mathcal{R}_{\rightarrow}(M)$ be the set of all indec reps. whose position in Γ_Q is in the slanted rectangular region w/ left boundary $\Sigma_{\rightarrow}(M)$

$\mathcal{R}_{\rightarrow}(M)$ is the <u>Maximal Slanted Rectangle</u> in Γ_Q w/ leftmost point M.

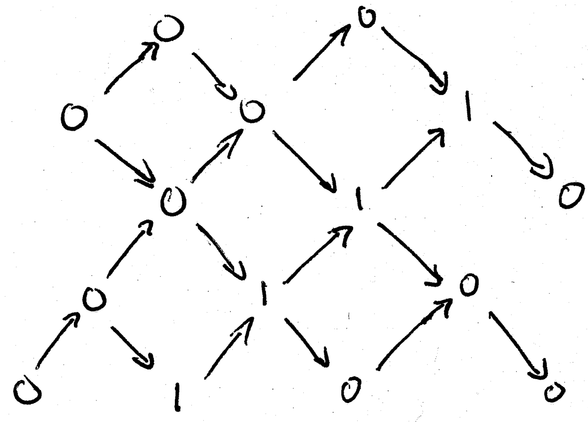
Then,
$$\dim(\text{Hom}(M,N)) = \begin{cases} 1 & N \in \mathcal{R}_{\rightarrow}(M) \\ 0 & N \notin \mathcal{R}_{\rightarrow}(M) \end{cases}$$

Continuing w/ same Γ_Q for $1 \leftarrow 2 \leftarrow 3 \rightarrow 4 \leftarrow 5$

we see the following dimension diagrams.



$\dim(\mathrm{Hom}(P(4), -))$



$\dim(\mathrm{Hom}(S(2), -))$

We can clearly see the leftmost 1 in each diagram corresponds to our choice of M. This is natural since $\mathrm{Hom}(M, M) = 1_M$ the identity is our basis of $\mathrm{Hom}(M, -)$.

We can also see that $\mathcal{R}_{\rightarrow}(P(i)) = \mathcal{R}_{\leftarrow}(I(i))$

For $\dim(\mathrm{Ext}^1(M, N))$ we adopt a few additional changes.

Assume $M \notin \mathrm{Proj}.\,Q$ since $\mathrm{Ext}^1(M, N) = 0$ if M proj.

Then $\Rightarrow \tau M \in \Gamma_Q$. A later result will ~~can~~ show

$$\mathrm{Ext}^1(M, N) \simeq D\,\mathrm{Hom}(N, \tau M)$$

w/ D the duality & τ the AR-translate. Then $\dim(\mathrm{Ext}^1(M, N))$ $= \dim(\mathrm{Hom}(N, \tau M))$ so we compute w/ $\mathcal{R}_{\leftarrow}(\tau M)$

Recall that elems. of $\mathrm{Ext}(M,N)$ can be represented through

$$0 \longrightarrow N \longrightarrow E \longrightarrow M \longrightarrow 0$$

w/ $E \in \mathrm{rep}\,Q$. We assume N, M indec. (E not necessarily indec.)

Want to find possible reps of E.

If $\dim(\mathrm{Ext}^1(M,N)) = 0 \Rightarrow E \simeq M \oplus N$.

If $\dim(\mathrm{Ext}^1(M,N)) = 1 \Rightarrow$ 1 other possibility of E up to isomorphism

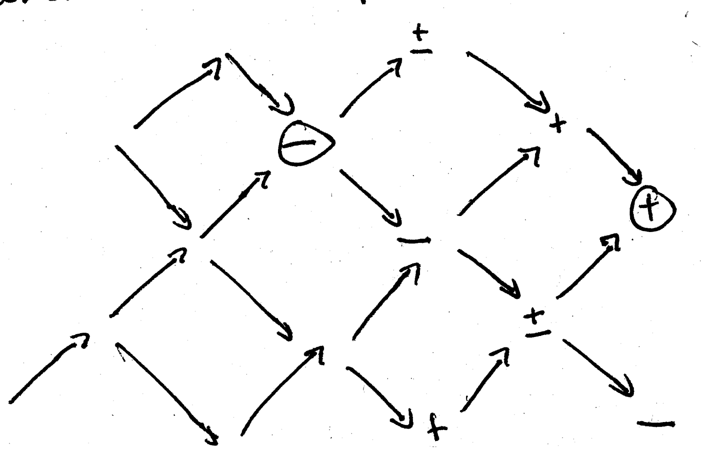
Let $M, N \in \mathrm{rep}\,Q$ indec. of Q in A_n type w/ $\mathrm{Ext}^1(M,N) \neq 0$

then

$N \in \mathcal{R}_{\leftarrow}(\tau M) \Rightarrow \Sigma_{\rightarrow}(N)$ & $\Sigma_{\leftarrow}(M)$ have 1 or 2 pts in common

These points in Γ_Q are the indecomposable summands of E.

Consider $0 \longrightarrow \begin{smallmatrix} 3 & & 5 \\ 2 & 4 & \\ 1 & & \end{smallmatrix} \longrightarrow \begin{smallmatrix} 3 & 5 \\ 4 & \end{smallmatrix} \oplus \begin{smallmatrix} 3 \\ 2 \\ 1 \end{smallmatrix} \longrightarrow 3 \longrightarrow 0$



Γ_Q diagram for above S.E. seq.

$\oplus = M$

$\ominus = N$

$+ \in \Sigma_{\leftarrow}(M)$

$- \in \Sigma_{\rightarrow}(N)$

3.2 Representation Type

A quiver Q is of __finite representation type__ if the number of isoclasses of indec reps of Q is finite.

The __underlying graph__ of Q is determined by replacing arrows $i \to j$ w/ edges $i - j$.

__Dynkin Diagrams__

$\mathbb{A}_n \quad 1 - 2 - \cdots - n-1 - n$

$\mathbb{B}_n \quad 1 = 2 - \cdots - n-1 - n$

$\mathbb{C}_n \quad 1 - 2 - \cdots - n-1 = n$

$$\mathbb{D}_n \quad \begin{array}{ccccccccc} & & & & & & & & n-1 \\ & & & & & & & \diagup & \\ 1 & - & 2 & - & \cdots & - & n-2 & & \\ & & & & & & & \diagdown & \\ & & & & & & & & n \end{array}$$

$$\mathbb{E}_6 \quad \begin{array}{ccccccccc} 1 & - & 2 & - & 3 & - & 4 & - & 5 \\ & & & & | & & & & \\ & & & & 6 & & & & \end{array}$$

$$\mathbb{E}_7 \quad \begin{array}{ccccccccccc} 1 & - & 2 & - & 3 & - & 4 & - & 5 & - & 6 \\ & & & & | & & & & & & \\ & & & & 7 & & & & & & \end{array}$$

$$\mathbb{E}_8 \quad \begin{array}{ccccccccccccc} 1 & - & 2 & - & 3 & - & 4 & - & 5 & - & 6 & - & 7 \\ & & & & | & & & & & & & & \\ & & & & 8 & & & & & & & & \end{array}$$

$\mathbb{F}_4 \quad 1 - 2 = 3 - 4$

$\mathbb{G}_2 \quad 1 \equiv 2$

Four infinite series; types $\mathbb{A}_n$, $\mathbb{B}_n$, $\mathbb{C}_n$ & $\mathbb{D}_n$

Five exceptional; $\mathbb{E}_6$, $\mathbb{E}_7$, $\mathbb{E}_8$, $\mathbb{F}_4$, $\mathbb{G}_2$

Types $\mathbb{A}_n$, $\mathbb{D}_n$, $\mathbb{E}_{6,7,8}$ are called __simply laced__ Dynkin diagrams.

This leads to (part) of the following result.

Gabriel's Theorem

A connected quiver is of finite representation type if and only if its underlying graph is Dynkin type $\mathbb{A}_n$, $\mathbb{D}_n$, $\mathbb{E}$.

Very powerful & surprising result. We prove it later.

Note that $\mathbb{E}$ has to have more than 5 vertices otherwise it is of type $\mathbb{D}_n$ or A_n. If $\mathbb{E}$ has 9 or more vertices then there are infinitely many indec. reps of $\mathbb{E}_{\geq 9}$.

Dynkin Diagrams show up in finite type classification in various fields

- Classifying Lie Algebras
- Root Systems
- Coxeter groups
- Cluster Algebras...

Note We omit section 3.3 "Auslander-Reiten Quivers of Type $\mathbb{D}_n$" as many of the ideas are similar to 3.1 & can be studied on the readers own.

3.4 Representations of Bound Quivers: Quivers w/ Relns.

Goal: Don't Limit the representations by limiting Q.

Allow for loops & cycles in the context of Relations

Defn. Let Q be a quiver

i) Two paths c, c' are parallel if $s(c)=s(c')$ & $t(c)=t(c')$

ii) A relation ρ is a lin. comb $\rho = \sum_c \lambda_c c$ of parallel paths (length ≥ 2)

iii) A bound quiver (Q, R) is a quiver Q w/ a set of relations $R := \{\rho\}$.

Defn. Let (Q, R) be a bound quiver. A representation of (Q, R) is a rep. of Q, $M = (M_i, \varphi_\alpha)$ s.t. $\varphi_\rho = 0$ for all $\rho \in R$,
where $\varphi_\rho = \sum_c \lambda_c \varphi_c$ if $\rho = \sum_c \lambda_c c$.

Naturally rep(Q, R) the category of reps. of (Q, R)

May easily define morphisms, sums, kernel/cokernel, etc...

$S(i)$ is defined the same as in rep Q

$P(i)$ & $I(i)$ we need the Path Algebra (Next chapter).

Ex. Let Q be

$$\begin{tikzcd} & 2 \arrow[dl, "\beta"'] & & \\ 1 \arrow[rr, "\gamma"'] & & 3 \arrow[ul, "\alpha"'] & 4 \arrow[l, "\delta"'] \end{tikzcd}$$

w/ $R = \{\alpha\beta, \beta\gamma, \gamma\alpha\}$. The following are the paths of (Q, R)

$$e_1, \gamma, e_2, \beta, e_3, \alpha, e_4, \delta, \delta\alpha$$

w/ indec. proj. reps given as

$$P(1) = \begin{matrix} 1 \\ 3 \end{matrix} \qquad P(2) = \begin{matrix} 2 \\ 1 \end{matrix} \qquad P(3) = \begin{matrix} 3 \\ 2 \end{matrix} \qquad P(4) = \begin{matrix} 4 \\ 3 \\ 2 \end{matrix}$$

rep(Q, R) is <u>not</u> a hereditary category as $S(3)$ above has min. proj. res

$$\cdots \longrightarrow \begin{matrix} 3 \\ 2 \end{matrix} \longrightarrow \begin{matrix} 1 \\ 3 \end{matrix} \longrightarrow \begin{matrix} 2 \\ 1 \end{matrix} \longrightarrow \begin{matrix} 3 \\ 2 \end{matrix} \longrightarrow 3 \longrightarrow 0$$

Not stopping after two steps.

We now work w/ Cluster-Tilted bound quivers of types $\mathbb{A}_n$ & $\mathbb{D}_n$.

Cluster-Tilted Bound Quivers of Type A_n

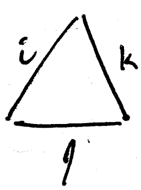
Recall polygonal/geometric interpretation of Γ_Q

$n+3$ regular gon

↳ Triangulations w/ property that each triangle has at least one side on the boundary of the polygon.

Cluster-Tilted quivers of A_n are associated w/ arbitrary triangulations of the $(n+3)$-gon.

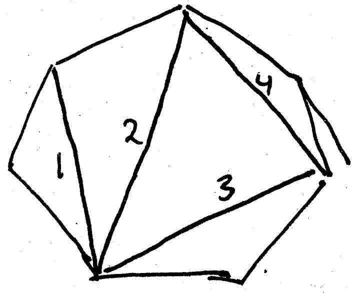
Let $T=\{1,2,\dots,n\}$ be a triang. on $n+3$-gon.
Define $Q=(Q_0,Q_1)$ by $Q_0=T$, $i\to j\in Q_1$ if i,j diag. bound a triangle in which j lies counterclockwise of i.



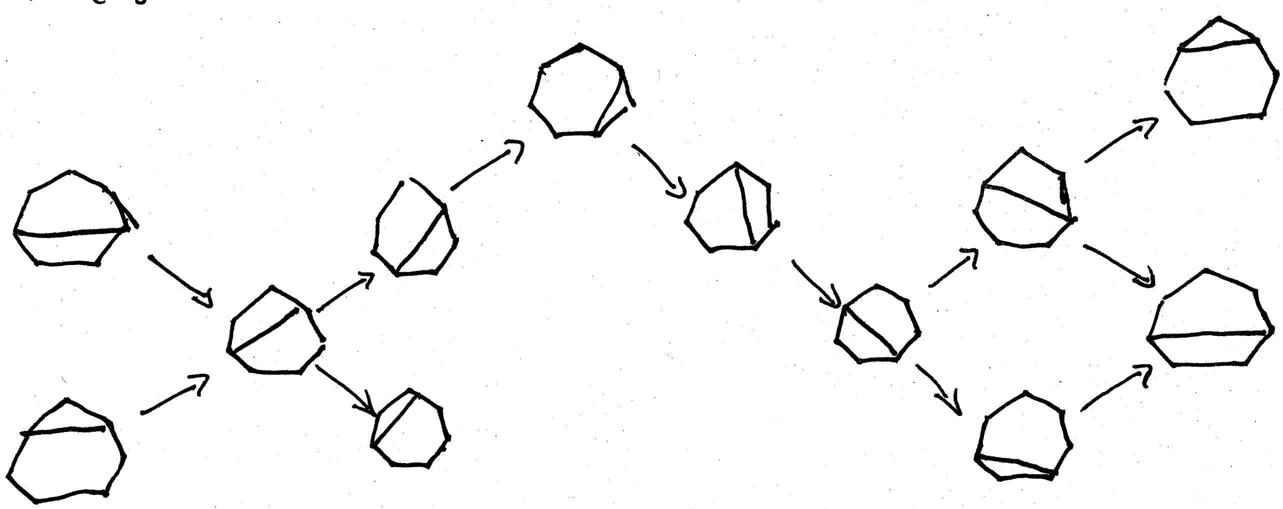
$$\rightsquigarrow \quad \begin{tikzcd} i \arrow[r] & j \arrow[r] & k \arrow[ll, bend right] \end{tikzcd}$$

Define the set $R:=\{\rho\}$ of relns. to be the set of all paths $i\to j\to k$ s.t. $\exists\; k\to i\in Q_1$. Then $\Gamma(Q,R)$ can be constructed as before.

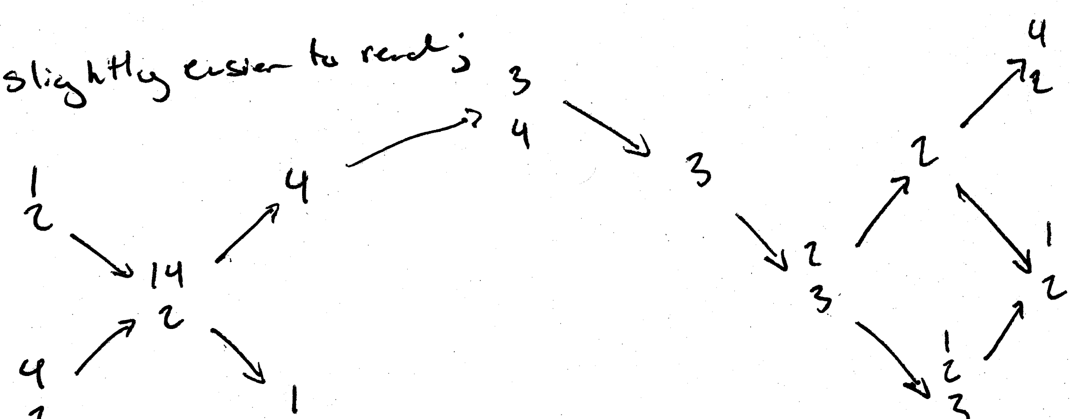
Ex.
Let Q: $\begin{tikzcd} 1 \arrow[r] & 2 \arrow[r] & 3 \arrow[dl] \\ & 4 \arrow[u] & \end{tikzcd}$ be associated w/ the following triang:



Then, $\Gamma_{(Q,R)}$ is



Or slightly easier to read;



(This is a mobius strip)

# of indec reps. of quivers of A_n is $\frac{n(n+1)}{2}$ $\left(\frac{n(n+3)}{2} - n\right)$

both w/ relns & not

Ch. 4. Algebras & Modules

(4.1) Ring Theory

Defn. A ring $R=(R,+,\cdot)$ is s.t. $(R,+)$ abelian grp & $(R,\cdot)$ closed associative, i.e. $(R,\cdot)$ is a semi-group (Magma w/ associativity).

Defn. A right-sided ideal (resp. left) I is a subgroup of $(R,+)$ s.t. $ar \in I$ $\forall\, a \in I, r \in R$. A two-sided ideal is an ideal that is right & left.

Exs.

a) $I=\{0\}$ is two-sided ideal in R

b) Let φ be a ring homo. then $\ker\varphi$ is two-sided ideal ($ar, ra \in \ker\varphi$)

c) Fix $a \in R$ then $aR:=\{ar \mid r\in R\}$ is a right ideal generated by a, ~~[illegible]~~

$Ra:=\{ra \mid r \in R\}$ is left-ideal generated by a.

$RaR:=\{ras \mid r,s \in R\}$ is two-sided-ideal generated by a.

d) Spse I two-sided ideal then R/I is quotient ring w/ mult $(a+I)\cdot(a'+I)=aa'+I$.

Defn. An ideal I is nilpotent if $I^m=0$ for some $m \geq 1$.

Defn. A proper Ideal $I \in R$ is maximal if for any ideal J s.t. $I \subset J \subset R$ s.t. $I=J$ or $J=R$.

Spse R commutative then I maximal iff R/I is a field

For k a field the only ideals are 0 & k.

<u>Defn.</u> The <u>Jacobson radical</u> rad R is the intersection of all maximal right ideals in R.

Zorn's Lemma $\Rightarrow$ $\text{rad } R \neq R$. Left radical $\equiv$ right radical

<u>Lemma</u> Let R be a ring and $a \in R$. TFAE:

1. $a \in \text{rad } R$
2. $\forall b \in R$, $1-ab$ has right inverse
3. $\forall b \in R$, $1-ab$ has two-sided inverse
4. $a \in \bigcap L$, where $\bigcap L$ is intersect of max. left ideals
5. $\forall b \in R$, $1-ba$ has left inverse
6. $\forall b \in R$, $1-ba$ has two-sided inverse.

<u>Pf.</u> $1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 1$ same arg for $4 \Rightarrow 5 \Rightarrow 6 \Rightarrow 4$. Then

$3 \Leftrightarrow 6$: Sps $1-ab$ has two-sided inverse c then

$$1 = 1-ba+ba = 1-ba+b(1-ab)ca = 1-ba+bca-babca = (1-ba)(1+bca)$$

Thus $(1+bca)$ ~~has~~ is right inverse of $1-ba$. Similarly,

$$1 = c(1-ab) \Rightarrow 1 = (1+bca)(1-ba)$$ thus $(1+bca)$ is left inverse of $1-ba$. $\square$

It follows that rad R is the intersection of all maximal left ideals in R.

Moreover rad R is a two-sided ideal in R.

Cor. $\operatorname{rad}\left(R/\operatorname{rad} R\right) = 0$

Pf. Sps $\mathfrak{I}$ two-sided ideal of R then $I \mapsto I/\mathfrak{I}$ is a bijection b/n ideals $I \in R$ which contain $\mathfrak{I}$ and the ideals in $R/\mathfrak{I}$. (sends max to max)

Then maximal ideals in $R/\operatorname{rad} R$ are of the form $I/\operatorname{rad} R$ for I maximal

Then $\operatorname{rad}(R/\operatorname{rad} R)$ is quotient of intersect of all max ideals of R.

That is $\operatorname{rad}(R/\operatorname{rad} R) = \operatorname{rad} R/\operatorname{rad} R = 0$. $\square$

Cor. If I two-sided nilpotent ideal in R, then $I \subset \operatorname{rad} R$.

Pf. I nilpotent $\Rightarrow$ $\forall x \in I$, $x^n = 0$. Then $\forall a \in R$ we have $(ax)^m = 0$

Thus $1 = 1-(ax)^m = (1+ax+(ax)^2+\cdots+(ax)^{m-1})(1-ax)$

Therefore $1-ax$ has a left inverse, Hence from above $x \in \operatorname{rad} R$ $\square$

$\Rightarrow I \subset \operatorname{rad} R$.

If R has only one maximal right ideal then R is a <u>local ring</u>

It's max. ideal is in fact $\operatorname{rad} R$.

4.2 Algebras

Recall k is algebraically closed field.

<u>Defn.</u> A k-algebra A is a ring $(A,+,\cdot)$ w/ unity 1 s.t. A has a k-vector space structure s.t.

1) addition in vector space A is the same as in ring A
2) scalar multiplication in v.s. A is compatible w/ ring multiplication, $\forall \lambda \in k$, $\forall a,b \in A$

$$\lambda(ab) = (\lambda a)b = a(\lambda b) = (ab)\lambda$$

Dimension of algebra A is dimension of vector space A.

<u>Exs.</u>

a) $k[x]$ polynomials of one indeterminate & coefs. in k.
b) $M_n(k)$, $n \times n$ matrices w/ entries in k. 1 is I_n (identity matrix $n \times n$)
c) If A is an algebra then the <u>opposite algebra</u> A^{op} defined the same way but multiplication rule ab in A^{op} is the same as ba in A.

Let $\beta = \{b_1, \ldots, b_n\}$ be a basis of A then every $a \in A$ is a lin. comb

Given $a, a' \in A$ we see

$$a a' = \sum_{i=1}^{n} \lambda_i b_i \sum_{j=1}^{n} \lambda'_j b_j = \sum_{i,j=1}^{n} \lambda_i \lambda'_j b_i b_j$$

Thus specifying how to multiply basis elements completely determines multiplication in the algebra A.

Sps $c=(i|\alpha_1,\dots,\alpha_r|j)$, $c'=(j|\alpha'_1,\dots,\alpha'_r|k)$ are two paths in Q s.t. $j=t(c)=s(c')$. The <u>concatenation</u> of paths $c\cdot c'$ is

$$c\cdot c'=(i|\alpha_1,\dots,\alpha_r,\alpha'_1,\dots,\alpha'_r|k).$$

<u>Defn</u> Let Q be a quiver. The <u>path algebra</u> kQ of Q is the algebra w/ basis all ~~elements~~ paths in Q. Multiplication is defined on basis elems c, c' as

$$cc'=\begin{cases} c\cdot c' & \text{if } s(c')=t(c) \\ 0 & \text{otherwise.}\end{cases}$$

Thus the product of two elems $\sum_{c,c'}\lambda_c\lambda^k_{c'}cc'$

<u>Lemma.</u> In path algebra kQ the unity element is the sum of constant paths

$$1=\sum_{i\in Q_0}e_i$$

<u>Pf.</u> Let $a\in A$. then $a=\sum_c\lambda_c c$ for sum $\lambda_c\in k$. Then $a\sum_{i\in Q_0}e_i=\sum_{i\in Q_0}\sum_c\lambda_c ce_i$ and ce_i is 0 if c doesn't end at i, $ce_i=c$ if path c ends at i. Thus $a\sum_{i\in Q_0}e_i=\sum_{i\in Q_0}\sum_{c\text{ st } t(c)=i}\lambda_c c=\sum_c\lambda_c c=a$. The other direction follows similarly. Thus, identity is as desired. $\square$

Examples

1. Let $Q = 1 \circlearrowleft \alpha$ the paths of Q are $e_1, \alpha, \alpha^2, \alpha^3, \ldots$

So kQ has basis $\{\alpha^t \mid t = 0, 1, 2, \ldots\} \Rightarrow kQ \simeq k[x]$

2. Let $Q = 1 \xrightarrow{\alpha_1} 2 \xrightarrow{\alpha_2} 3 \xrightarrow{\alpha_3} \cdots \xrightarrow{\alpha_{n-1}} n$

The basis elems are

$$\begin{array}{ccccc} e_1 & \alpha_1 & \alpha_1\alpha_2 & \cdots & \alpha_1\alpha_2\alpha_3\cdots\alpha_{n-1} \\ & e_2 & \alpha_2 & \cdots & \alpha_2\alpha_3\cdots\alpha_{n-1} \\ & & e_3 & \ddots & \vdots \\ & & & & e_n \end{array}$$

Thus, $kQ \simeq$ Algebra of upper triangular $n \times n$ matrices.

<u>Defn.</u> If A & B are k-algebras then a k-linear map $f: A \to B$ is a <u>homomorphism of k-algebras</u> if $f(1) = 1$ and

$$f(aa') = f(a) f(a')$$

<u>Defn.</u> Let B be a k-vector subspace of A. Then B is a <u>subalgebra</u> if B contains unity elem. 1 & $\forall\, b, b' \in B$ $bb' \in B$.

The only ideal which is a subalgebra is $A = I$.

subalgebra $\Rightarrow 1 \in I$, I ideal $\Rightarrow$ $1a = a \in I$ $\forall a \in A$.

<u>Prop.</u> If I is 2-sided ideal nilpotent in A s.t. the algebra $A/I \simeq k \times \cdots \times k$. Then $I = \operatorname{rad} A$.

<u>Pf.</u> Previously we have $I \subset \operatorname{rad} A$. Suffices to show $\operatorname{rad} A \subset I$.

k a field $\Rightarrow$ only ideals are 0 & k. Thus only ideals in $k \times \cdots \times k$

$0 \times k \times \cdots \times k$, $k \times 0 \times \cdots \times k$, $\ldots$, $k \times k \times \cdots \times 0$

Thus $\operatorname{rad}(A/I) = 0$.

Consider $\pi : A \to A/I$, $\pi(a) = a + I$, $a \in \operatorname{rad} A$.

For every $b \in A$, $1 - ba$ has two-sided inverse $c \in A$.

$1 + I = \pi(1) = \pi(c(1-ba)) = \pi(c)\,\pi(1-ba) = \pi(c)(1 - \pi(b)\pi(a))$

Therefore $1 - \pi(b)\pi(a)$ has left inverse in A/I.

Thus $\pi(a) \in \operatorname{rad}(A/I) = 0 \Rightarrow a \in I \Rightarrow \operatorname{rad} A \subset I$. $\square$

<u>Cor.</u> If Q is a quiver w/o orientated cycles, then $\operatorname{rad} kQ$ is the two-sided ideal generated by all arrows in Q.

<u>Pf.</u> R_Q ideal generated by arrows, any prod of $l+1$ arrows $= 0$

$R_Q^{l+1} = 0$, R_Q nilpotent. kQ/R_Q has basis $\{e_i + R_Q \mid i \in Q_0\}$

$kQ/R_Q \simeq k \times \cdots \times k$. Follows from above prop. $\square$

4.3 Modules

<u>Defn.</u> Let R be a ring w/ $1 \neq 0$. A right R-Module M is an abelian group w/ binary operation (right R-action)

$$M \times R \longrightarrow M \qquad (m, r) \longmapsto mr$$

s.t. $\forall\, m_1, m_2 \in M$ and $r_1, r_2 \in R$ we have

1) $(m_1 + m_2) r = m_1 r + m_2 r$
2) $m_1 (r_1 + r_2) = m_1 r_1 + m_1 r_2$
3) $m_1 (r_1 r_2) = (m_1 r_1) r_2$
4) $m_1 1 = m_1$

<u>Examples</u>

1. ring R is an R-module given by multiplication in R
2. If $I < R$ right ideal then I is an R-module $aR := \{ar \mid r \in R\}$
3. If $I < R$ & M an R-module then $MI := \{m_1 r_1 + \cdots + m_t r_t \mid m_i \in M, r_i \in I\}$ is a <u>submodule</u> of M.
4. If A k-algebra then any A-module M is also a k-vector space. <u>underlying vector space</u> of A-module M, $m\lambda = m(\lambda 1_A)$ for $m \in M$ & $\lambda \in k$.
5. Let $A = kQ$ the path algebra. For each $i \in Q_0$ the module $S(i)$ w/ basis $\{e_i\}$ ($S(i)$ is one-dimensional) & A-module structure is given as

$$m e_i c := \begin{cases} m e_i & \text{if } c = e_i \\ 0 & \text{otherwise} \end{cases}$$

6. Let $kQ = A$ be a path algebra. For each $i \xrightarrow{\alpha} j \in Q_1$ we define the A-module $M(\alpha)$ w/ vector space k^2, Basis given as $\{e_i, \alpha\}$ & A-module struct. given as

$$(\lambda_i e_i + \lambda_\alpha \alpha)c = \lambda_i e_i c + \lambda_\alpha \alpha c = \begin{cases} \lambda_i e_i & \text{if } c = e_i \\ \lambda_\alpha \alpha & \text{if } c = e_j \\ \lambda_i \alpha & \text{if } c = \alpha \\ 0 & \text{otherwise} \end{cases}$$

<u>Defn.</u> A module M is said to be <u>generated by the elements</u> $m_1, m_2, \ldots, m_s$ if $\forall m \in M$ there exist $a_i \in R$ s.t.

$$m = m_1 a_1 + \cdots + m_s a_s$$

M is <u>finitely generated</u> if is generated by the finite set of elems.

The ideal aR is an R-module generated by one elem. a.

<u>Defn.</u> Let M, N be two R-modules. A map $h: M \longrightarrow N$ is a <u>morphism of R-modules</u> if $\forall m, m' \in M$ & $a \in R$

$$h(m + m') = h(m) + h(m') \quad \text{and} \quad h(ma) = h(m)a$$

$\ker h := \{m \in M \mid h(m) = 0\}$, $\operatorname{im} h := \{h(m) \mid m \in M\}$, $\operatorname{coker} h := N/\operatorname{im} h$

Can see that kernel, image, coker are all A-modules as well.

Ex. Let $A = kQ$ w/ $S(j)$ & $M(\alpha)$ as previous.

Then $h: S(j) \longrightarrow M(\alpha)$ is a morphism,

$h(me_j + m'e_j) = h((m+m')e_j) = (m+m')\alpha = m\alpha + m'\alpha = h(me_j) + h(m'e_j)$

and $h(\lambda me_j) = \lambda m\alpha = \lambda h(me_j)$ so h is k-linear.

$h(me_j e_j) = h(me_j) = m\alpha = m\alpha e_j = h(me_j)e_j$

If c is a path besides e_j then $h(me_j c) = h(0) = 0 = m\alpha c = h(me_j)c$

Therefore h is a morphism of A-modules.

Ex. Let A be a k-algebra and M an A-module.

An <u>endomorphism</u> $f: M \longrightarrow M$ is a morphism of A-modules.

The set of all endomorphisms of M is $\operatorname{End} M$.

Naturally, $\operatorname{End} M$ has k-vector space structure.

Moreover, $\operatorname{End} M$ is an algebra w/ multiplication given as composition of morphisms.

We now shift to a few results beginning w/ Nakayama's Lemma.

<u>Nakayama's Lemma</u> Let M be a finitely-generated R-module and I be a two-sided ideal in R s.t. I is contained in rad R. If $MI = M$ then $M = 0$.

<u>Pf.</u> (induction on size of generating set of M)

Let M be generated by $\{m_1, \ldots, m_s\}$ & $MI = M$.

Sps $s = 1$ then $M = MI$ implies $m_1 = m_1' r_1 + \cdots + m_t' r_t$ for some $m_i' \in M$ and $r_i \in I$. Then $M = m_1 R$ it follows that $\exists\, a_i \in R$ s.t. $m_i' = m_1 a_i$ $\forall i$. Let $x = a_1 r_1 + \cdots + a_t r_s \in I$ so $m_1 = m_1 x$ $\Rightarrow m_1(1-x) = 0$ but $x \in I \subset \text{rad } R \Rightarrow (1-x)$ has two-sided inv. b. So $0 = m_1(1-x)b = m_1$ and since m_1 generates M we get $M = 0$. BASE CASE DONE.

Sps $s \geq 2$ then $M = MI \Rightarrow \exists\, m \in M,\ x \in I$ s.t. $m_1 = mx$

But $M = \langle m_1, \ldots, m_s \rangle \Rightarrow \exists\, a_i$ s.t. $m = m_1 a_1 + \cdots + m_s a_s$.

Therefore $m_1 = m_1 a_1 x + m_2 a_2 x + \cdots + m_s a_s x$

$\Rightarrow \quad m_1(1 - a_1 x) = m_2 a_2 x + \cdots + m_s a_s x$

But $x \in I \subset \text{rad } R \Rightarrow (1 - a_1 x)$ has two-sided inv. b. So

$m_1 = m_2 a_2 x b + \cdots + m_s a_s x b \Rightarrow R$ is generated by the $s-1$ elements $m_2, \ldots, m_s$

By induction it follows that $M = 0$. $\square$

Cor. If A is finite-dimensional algebra, then $\operatorname{rad} A$ is nilpotent.

Pf. A finite-dim. $\Rightarrow$ All ideals in A finite dim. (they have k-basis) thus ideals of A are finitely-generated A-modules. So we get the chain

$$A \supset \operatorname{rad} A \supset (\operatorname{rad} A)^2 \supset (\operatorname{rad} A)^3 \supset \cdots$$

which becomes stationary, i.e., $(\operatorname{rad} A)^n = (\operatorname{rad} A)^m$ $\forall\, n \geq m$. That is $(\operatorname{rad} A)^m = (\operatorname{rad} A)^m (\operatorname{rad} A)$ by Nakayama's Lemma $\Rightarrow (\operatorname{rad} A)^m = 0$. $\square$

Five Lemma Given commutative diagram of R-modules w/ exact rows

$$\begin{array}{ccccccccc}
M_1 & \xrightarrow{f_1} & M_2 & \xrightarrow{f_2} & M_3 & \xrightarrow{f_3} & M_4 & \xrightarrow{f_4} & M_5 \\
\downarrow{\scriptstyle \phi_1} & & \downarrow{\scriptstyle \phi_2} & & \downarrow{\scriptstyle \phi_3} & & \downarrow{\scriptstyle \phi_4} & & \downarrow{\scriptstyle \phi_5} \\
N_1 & \xrightarrow{g_1} & N_2 & \xrightarrow{g_2} & N_3 & \xrightarrow{g_3} & N_4 & \xrightarrow{g_4} & N_5
\end{array}$$

Then,

1) ϕ_2, ϕ_4 surj. & ϕ_5 inj. $\Rightarrow \phi_3$ surj.
2) ϕ_1 surj. & ϕ_2, ϕ_4 inj. $\Rightarrow \phi_3$ inj.
3) ϕ_1 surj. & ϕ_2, ϕ_4 isomorphisms, & ϕ_5 inj. $\Rightarrow \phi_3$ isomorphism.

Pf. We prove 1), 3) follows from 1) & 2)

Let $n_3 \in N_3$, ϕ_4 surj. $\Rightarrow \exists\, m_4 \in M_4$ s.t. $\phi_4(m_4) = g_3(n_3)$

2nd row exact $\Rightarrow g_4 g_3(n_3) = 0 \Rightarrow 0 = g_4 \phi_4(m_4) = \phi_5 f_4(m_4)$ by commutativity. ϕ_5 inj. $\Rightarrow f_4(m_4) = 0$ 1st row exact $\Rightarrow \exists\, m_3 \in M_3$ s.t. $f_3(m_3) = m_4$.

Pf. of Five Lemma cont.

We now have

$$g_3\phi_3(m_3) = \phi_4 f_3(m_3) = \phi_4(m_4) = g_3(n_3)$$

Hence $g(n_3 - \phi_3(m_3)) = 0$. 2^{nd} row exact $\Rightarrow \exists\, n_2 \in N_2$ s.t. $g_2(n_2) = n_3 - \phi_3(m_3)$. ϕ_2 surj. $\Rightarrow \exists\, m_2 \in M_2$ s.t. $\phi_2(m_2) = n_2$. $n_3 - \phi_3(m_3) = g_2\phi_2(m_2) = \phi_3 f_2(m_2)$ since diag. commutative. Then,

$$\phi_3(f_2(m_2) + m_3) = \phi_3 f_2(m_2) + \phi_3(m_3) = n_3$$

So $n_3 \in \operatorname{Im}\phi_3$ thus ϕ_3 surjective. $\square$

4.4 Idempotents & Direct Sum Decomposition.

Let A be a k-algebra.

Defn. Let $M_1, \ldots, M_s$ be A-modules. Then the direct sum $M_1 \oplus M_2 \oplus \cdots \oplus M_s$ is an A-module w/ vector space as direct-sum of vector spaces of M_i. And module structure is given as

$$(m_1, m_2, \ldots, m_s)a = (m_1 a, m_2 a, \ldots, m_s a)$$

A module is called indecomposable if it cannot be written as the direct sum of two proper submodules.

Goal: Given an Algebra A (which itself is an A-module) we wish to give a direct sum decomp. of the A-module A into indecomposable parts.

Defn. An element $e \in A$ is idempotent if $e^2 = e$. Two idempotents are orthogonal if $e_1 e_2 = e_2 e_1 = 0$. $e \in A$ is central idempotent if $ea = ae$ $\forall a \in A$. $e \neq 0 \in A$ is called primitive if e cannot be written as $e = e_1 + e_2$ for nonzero orthogonal idempotents.

$0, 1$ are trivial idempotents.

Lem: If $A = kQ$ is a path alg. then each constant path e_i is a primitive idempotent.

Pf. It follows immediately that e_i is an idempotent.

Sps $e_i = e + e'$ w/ $e, e' \in A$ orthogonal idempotents, $e = \sum \lambda_c c$, $e' = \sum \lambda'_c c$

Since $e = \sum \lambda_c c$ is an idempotent then

$$0 = e^2 - e = \sum \lambda_c \lambda_{c'} cc' - \sum \lambda_{c''} c'' = \sum_{cc' = c''} (\lambda_c \lambda_{c'} - \lambda_{c''}) c''.$$

So $\lambda_{e_j} \lambda_{e_j} = \lambda_{e_j} \Rightarrow \lambda_{e_j} = 0$ or $\lambda_{e_j} = 1$ $\forall j \in Q_0$.

But $ee' = 0 \Rightarrow \lambda'_{e_j} = 0$ for $\lambda_{e_j} = 1$ and $e + e' = e_i \Rightarrow$ if $i \neq j$

$\lambda_{e_j} = 0$ & $\lambda'_{e_j} = 0$ and one of $\lambda_{e_i}, \lambda'_{e_i}$ is 0 and the other is 1.

Sps. WLOG $\lambda_{e_i} = 1$ & $\lambda'_{e_i} = 0$ it follows that for any path c, $\lambda_{c'} = 0$ thus $e' = 0$ and e_i is primitive. $\square$

Lem. Let e be a non-trivial idempotent. Then e and $(1-e)$ are orthogonal idempotents s.t. $1 = e + (1-e)$ and the right A-module A is given as (equal to)

$$A = eA \oplus (1-e)A.$$

If e is central, then $A = eA \oplus (1-e)A$ as K-algebras.

Pf. $(1-e)^2 = 1 - e - e + e^2 = 1 - e - e + e = 1 - e \Rightarrow (1-e)$ idempotent.

$e(1-e) = e - e^2 = e - e = 0$ And $(1-e)e = e - e^2 = e - e = 0$

$\Rightarrow e, (1-e)$ orthogonal. Consider A, let $a \in A$ then

$a = ea + a - ea = ea + (1-e)a \in eA + (1-e)A \Rightarrow A = eA + (1-e)A$

to show this is direct sum, suffices to show $eA \cap (1-e)A = \{0\}$.

Sps ~~$a \in A$ st~~ $a \in eA \cap (1-e)A$ then $a = ea' = (1-e)a''$ for $a', a'' \in A$

$ea' - (1-e)a'' = 0 \Rightarrow 0 = e0 = e(ea' - (1-e)a'') = ea' \Rightarrow 0 = ea' = a$

So $eA \cap (1-e)A = \{0\} \Rightarrow A = eA \oplus (1-e)A$.

This respects A-module struct. since $ab = (ea + (1-e)a)b = eab + (1-e)ab$.

If e is central then $eA \oplus (1-e)A$ is a K-alg w/ componet. mult.

$(ea, (1-e)a) \cdot (ea', (1-e)a') = (eaea', (1-e)a(1-e)a')$

Since $eaea' = eaa'$ then $(1-e)a(1-e)a' = (1-e)aa' - (1-e)eaa' = (1-e)aa'$. $\square$

Ex. Let $Q = 1 \xrightarrow{\alpha} 2$ then $A = kQ$.

We have e_1 & $e_2 = 1 - e_1$ are orthogonal idempotents thus A decomposes as $A = e_1 A \oplus e_2 A$. e_1 is not central since $e_1 \alpha = \alpha$ but $\alpha e_1 = 0$. So $e_1 A \oplus e_2 A$ is not reflective of the structure of A. For any $e_2 a \in e_2 A$ we have $\alpha(e_2 a) \in e_1 A$.

The above result shows that orthogonal idempotents (w/ sum $= 1$) lead to a direct sum decomp. of A-module. The following shows the opposite direction — direct sum decomp. $\Rightarrow$ orthogonal idemp. w/ sum $= 1$.

Lem. Let $A = M_1 \oplus M_2$ be a direct sum decomp. of the A-module A.
Then,
1) $\exists\, e_1 \in M_1$, $e_2 \in M_2$ s.t. e_1, e_2 orthogonal idemp. and $e_1 + e_2 = 1$.
2) M_i is indecomp. iff e_i is primitive for $i = 1, 2$.

Sps A finite-dim then $A = M_1 \oplus \cdots \oplus M_n$ w/ M_i indecomp. A-modules.
$\Rightarrow \exists$ primitive, pairwise orthogonal idempotents $e_1, e_2, \ldots, e_n$ s.t.
$M_i = e_i A$ and $e_1 + e_2 + \cdots + e_n = 1$.

The converse statement holds as well.

4.5 Criterion for Indecomposability

<u>Goal:</u> Relate endomorphism alg. to module indecomposability.

<u>Defn.</u> An algebra A is <u>local</u> if A has a unique max (right) ideal.

A local $\Rightarrow$ rad $A = I$ for unique max ideal I.

<u>Lem.</u> Let A be a k-alg. The following are equivalent:

1) A is local
2) A has unique max left ideal
3) Set of non-invertible elems. of A is a two-sided ideal
4) $\forall a \in A$ we have a or $(1-a)$ is invertible
5) The k-alg. $(A/\text{rad } A)$ is a field.

<u>Pf.</u>

$(1 \Rightarrow 3)$ A local $\Rightarrow$ rad $A = I$, I proper has no invertible elems.
if x not invertible $\Rightarrow x \in \text{rad } A$ since $\langle x \rangle \neq A \Rightarrow \langle x \rangle \subset \text{rad } A$.
Thus non-invertible elems of A = rad A (which is two-sided ideal).

$(2 \Rightarrow 3)$ Follows.

$(3 \Rightarrow 4)$ Sps a, $(1-a)$ are non-invertible then $1 = a + (1-a)$ is non-invertible
which is a contradiction (1 is invertible).

$(4 \Rightarrow 5)$ NTS $\forall a \neq 0 \in (A/\text{rad } A)$ is invertible. That is $\forall a \in (A \setminus \text{rad } A)$
then is some $c \in A$ s.t. $(1-ac) \in \text{rad } A$. Since $a \notin \text{rad } A$ we have $\exists b$
s.t. $(1-ab)$ has no inverse in A. Then $(4) \Rightarrow ab$ has inverse b'.
Hence $1 = abb'$ so $c = bb'$ and the result follows.

$(5 \Rightarrow 1, 2)$ $(5) \Rightarrow$ rad A is max two-sided ideal (since $A/\text{rad } A$ is a field)
Thus $(5) \Rightarrow (1)$ & $(5) \Rightarrow (2)$. $\square$

Cor. If A finite-dim local k-alg then $(A/\operatorname{rad} A) \simeq k$.

Pf. Follows that $(A/\operatorname{rad} A)$ is a field ext. of k.
A finite-dim $\Rightarrow$ the ext. is finite dim. $\Rightarrow$ algebraic ext.
Result follows since k alg. closed. $\square$

To see why k has to be alg. closed observe, $k = \mathbb{R}$ and $A = \mathbb{C}$
where $\mathbb{C}$ is an $\mathbb{R}$-algebra. Obviously $(\mathbb{C}/\operatorname{rad}\mathbb{C}) \not\simeq \mathbb{R}$.

Cor. If A is local then A only has trivial idempotents 0 & 1.

Pf. Sps $e \in A$ idempotent. Then $e(1-e) = 0$ but this implies
that e or $(1-e)$ is invertible $\Rightarrow$ Thus, $e = 0$ or $(1-e) = 0$. $\square$

Cor. An idempotent $e \in A$ is primitive iff the alg. eAe has only
trivial idempotents 0 & 1.

Pf. $(\Rightarrow)$ Observe $e = 1$ in eAe.
Let e be prim. idemp. $\in A$, $e' \in eAe$ idemp. $\Rightarrow$ $e' = eae$ for some $a \in A$
then $(e-e')$ is an idemp. and $e'(e-e') = 0$ $\Rightarrow$ $e = e' + (e-e')$
e prim. $\Rightarrow$ $e' = 0$ or $e - e' = 0$. This shows $e - e' = 0 \Rightarrow e'$ is trivial
idemp.; hence all idemp. of eAe are trivial.

$(\Leftarrow)$ Sps $e = e' + e''$ for e', e'' orthogonal idemp. in A. Then
$(ee'e)(ee'e) = ee'eee'e = ee'(e'+e'')e'e = ee'e + ee'e''e'e = ee'e \qquad (e'e'' = 0)$
Thus $(ee'e) \in A$ idemp. and hence idemp. in eAe. Follows that $ee'e = 0$ or $= e$
$0 = ee'e = (e'+e'')e'(e'+e'') = e' \left[e'e'' = e''e' = 0\right]$ and $e' + e'' = e = ee'e = (e'+e'')e'(e'+e'') = e'$
Thus $e'' = 0$ $\Rightarrow$ e <u>can't</u> be written as two non-trivial orth. idemp. $\Rightarrow e$ primitive. $\square$

Cor. Let A be a k-algebra, let M be a finite-dim A-module;

Let $\operatorname{End} M$ be its endomorphism algebra. TFAE:

1) M is indecomposable
2) Every endomorphism $f \in \operatorname{End} M$ is of the form $f = \lambda 1_M + g$ w/ $g \in \operatorname{End} M$ s.t. g nilpotent and $\lambda \in k$.
3) $\operatorname{End} M$ is local.

Pf. $(1 \Rightarrow 2)$ Let $f \in \operatorname{End} M$, $f: M \to M$ is a k-linear map b/n finite-dim k-vector space. k alg. closed $\Rightarrow$ char. poly of f is expressed as

$\chi_f(x) = \prod_{i=1}^{t} (\lambda - \lambda_i)^{v_i} \Rightarrow \lambda_i$ are eigenvalues of f and by the spectral thm. $\exists$ basis β of M s.t. $(f)_\beta$ is triangular matrix w/ diagonal entries are eigenvalues λ_i w/ multiplicities v_i.

Let $M_i := \ker(f - \lambda_i 1_M)^{v_i}$ so $\dim M_i = v_i$ and $M = M_1 \oplus \cdots \oplus M_t$. $(*)$

Let $h_i := (f - \lambda_i 1_M)^{v_i}$ so h_i is a poly. in f. That is for some $a \in k$,

$$h_i = f^{v_i} + a_{v_i - 1} f^{v_i - 1} + \cdots + a_1 f + a_0 1_M.$$

Since $f \in \operatorname{End} M \Rightarrow h_i \in \operatorname{End} M \Rightarrow \ker h_i = M_i$ is an A-module. So $(*)$ is a direct sum decomp. of A into A-modules. M indec. $\Rightarrow t = 1$

thus f only has one eigenvalue λ. Thus $(f)_\beta$ is a triangular matrix w/ diag entries all given as λ, thus $f = \lambda 1_M + g$ w/ g nilpotent.

Pf. cont. $(2 \Rightarrow 3)$ Let $f = \lambda 1_M + g \in \operatorname{End} M$.
If f not invertible then $\lambda = 0$ and $f = g$ is nilpotent. Thus,
there exists $l \geq 0$ s.t. $f^l = 0$, but

$$1_M = 1_M - f^l = (1_M + f + f^2 + \cdots + f^{l-1})(1-f)$$

so $(1-f)$ is invertible, and previous result implies $\operatorname{End} M$ is local.

$(3 \Rightarrow 1)$ Assume $\operatorname{End} M$ is local and sps $M = M_1 \oplus M_2$.
Let $p_i : M \to M_i$ be the canonical proj. and $u_i : M_i \to M$ the canonical inj.
Then $u_i \circ p_i \in \operatorname{End} M$ and $(u_i \circ p_i)^2 = u_i \circ p_i \Rightarrow u_i \circ p_i$ is an
idemp. in $\operatorname{End} M \Rightarrow u_i \circ p_i = 0$ or $u_i \circ p_i = 1$ since $\operatorname{End} M$ local.
If $u_i \circ p_i = 0$ then $M_i = 0$. If $u_i \circ p_i = 1$ then $M_i = M$.
That is, M is indecomposable. $\square$

We show 2 exs of endomorphism algebras, representing modules of kQ
as reps in $\operatorname{rep} Q$. (The two categories $\operatorname{rep} Q = \operatorname{mod} kQ$).

<u>Ex.</u> Let Q = Dynkin type $\mathbb{A}_n$, $\mathbb{D}_n$, or $\mathbb{E}_{6,7,8}$. From comp. in Ch 3.
we see that if M is indecomposable rep of Q then $\operatorname{End} M = k$ and k is local.

If M is not indecomp. then the identity morphism on each indecomp. summand
of M is a <u>non-trivial</u> idempotent in $\operatorname{End} M$. $\Rightarrow \operatorname{End} M$ is <u>not local</u>.

<u>Ex.</u> Let Q be the Kronecker 2 quiver; $1 \underset{\beta}{\overset{\alpha}{\rightrightarrows}} 2$

w/ representation M given directly as:

$$k^2 \underset{\begin{bmatrix} 1 & \lambda \\ 0 & 1 \end{bmatrix}}{\overset{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}{\rightrightarrows}} k^2$$

Then an endomorphism of M is a morphism of reps. $f: M \to M$ (two lin. maps $f_1, f_2 : k^2 \to k^2$)

s.t. $f_1, f_2 : k^2 \to k^2$ commute w/ maps $\varphi_\alpha, \varphi_\beta$ of the rep. M.

Since $\varphi_\alpha = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2 = id : k^2 \to k^2$ we see f_1 & f_2 have the same matrix.

Say they are given as

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Since $f_2 \varphi_\beta = \varphi_\beta f_1$ we have $\begin{bmatrix} a & \lambda a + b \\ c & \lambda c + d \end{bmatrix} = \begin{bmatrix} a + \lambda c & b + \lambda d \\ c & d \end{bmatrix}$

If $\lambda \neq 0$ this implies that $c = 0$ and $a = d$ thus $End M := \left\{ \begin{bmatrix} a & b \\ 0 & a \end{bmatrix} \middle| a, b \in k \right\}$

Then $x \in End M$ invertible iff $a \neq 0$, if $a = 0$ then $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} a & b \\ 0 & a \end{bmatrix}$ is invertible. Thus $End M$ is local $\Rightarrow M$ indecomposable.

If $\lambda = 0$ then $End M \cong M_2(K)$ which is <u>not</u> local since

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

are not invertible (both).

Ch. 8 Quadratic Forms & Gabriel's Thm.

Goal: Prove Gabriel's Theorem.

8.1 Variety of reps.

Let Q be a quiver w/o orientated cycles. Define $\underline{d} = (d_i) \in \mathbb{Z}^n$ as the dimension vector of Q.

Let $E_{\underline{d}}$ be the space of all reps. $M = (M_i, \varphi_\alpha)_{i \in Q_0, \alpha \in Q_1}$ w/ dim. vector $\underline{d}$
$\Rightarrow M_i \simeq k^{d_i}$, so the M_i reps of $E_{\underline{d}}$ are fixed up to isomorphism and the reps are determined by their linear maps φ_α. So $E_{\underline{d}} \simeq \bigoplus_{\alpha \in Q_1} \mathrm{Hom}_k(k^{d_{s(\alpha)}}, k^{d_{t(\alpha)}})$

where $\mathrm{Hom}_k(k^{d_{s(\alpha)}}, k^{d_{t(\alpha)}}) \simeq M_{d_{t(\alpha)} \times d_{s(\alpha)}}(k)$

So $\dim(E_{\underline{d}}) = \sum_\alpha d_{s(\alpha)} d_{t(\alpha)}$ (E_d is k-vector space).

Let $G_{\underline{d}} := \prod_{i \in Q_0} GL_{d_i}(k)$ be a group. $G_{\underline{d}}$ acts on $E_{\underline{d}}$ by conjugation.

if $g = (g_i) \in G_{\underline{d}}$, $M = (M_i, \varphi_\alpha) \in E_{\underline{d}}$, and $i \xrightarrow{\alpha} j \in Q$ then

$$(g \cdot \varphi)_\alpha = g_j \varphi_\alpha g_i^{-1} : \quad \begin{tikzcd} k^{d_i} \arrow[r, "\varphi_\alpha"] \arrow[loop below, "g_i"] & k^{d_j} \arrow[loop below, "g_j"] \end{tikzcd}$$

Denote the orbit of a rep M under this action by $\mathcal{O}_M$.

$$\mathcal{O}_M := \{ g \cdot M \mid g \in G_{\underline{d}} \}$$

<u>Lem.</u> The orbit $\mathcal{O}_M$ is the isoclass of the rep M. That is,

$$\mathcal{O}_M := \{ M' \in \operatorname{rep} Q \mid M' \cong M \}$$

<u>Pf.</u> Sps $M = (M_i, \varphi_\alpha)$, $M' = (M'_i, \varphi'_\alpha)$ are in the same orbit.

$\Rightarrow \exists g = (g_i)_{i \in Q_0}$ s.t. $g \cdot M = M' \Rightarrow \forall \alpha \in Q_1$, $\alpha : i \to j$

$$\begin{tikzcd} M_i \arrow[r, "\varphi_\alpha"] \arrow[d, "g_i"'] & M_j \arrow[d, "g_j"] \\ M'_i \arrow[r, "\varphi'_\alpha"'] & M'_j \end{tikzcd}$$

commutes. $\Rightarrow$ g is a morphism of reps and $g_i \in GL_{d_i}(k)$

$\Rightarrow g_i$ invertible $\forall i \in Q_0 \Rightarrow g$ is an <u>isomorphism</u> of reps. $(M \cong M')$

Sps now that $g : M \to M'$ is an iso. then each $g_i \in GL_{d_i}(k)$
and $M' = g(M) = g \cdot M$ as desired. $\square$

The <u>stabilizer</u> $\operatorname{Stab} M := \{ g \in G_{\underline{d}} \mid g \cdot M = M \}$ of the rep M.
corresponds w/ the Automorphism (group) $\operatorname{Aut} M$ of the rep M.

We have the following Algebraic Geometry facts:

<u>Lem.</u> Let $\underline{d} \in \mathbb{Z}^n_{\geq 0}$ then

1) For any rep. M of dim. vector $\underline{d}$ the dimensions satisfy

$$\dim \mathcal{O}_M = \dim G_{\underline{d}} - \dim \operatorname{Aut} M$$

2) There is at most one orbit of $\mathcal{O}_M$ of codim. zero in $E_{\underline{d}}$.

Lem. If $0 \to L \xrightarrow{f} M \xrightarrow{g} N \to 0$ is a (non-split) short exact sequence of reps, then $\dim \mathcal{O}_{L\oplus N} < \dim \mathcal{O}_M$.

Pf. Let $L = (L_i, \psi_\alpha)$, $M = (M_i, \varphi_\alpha)$, $N = (N_i, \chi_\alpha)$

For each $i \in Q_0$ let B_i' be a basis of L_i, extend $f_i(B_i')$ to a basis of M_i called B_i, extend $g_i(B_i)$ to a basis B_i'' of N_i. WRT these bases we have

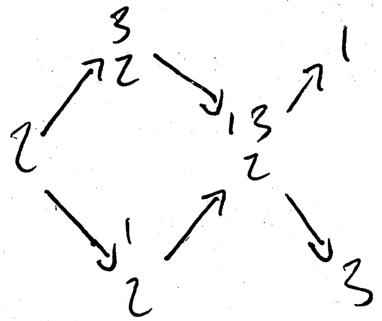
$$f_i = \left[\begin{array}{c} \begin{matrix}1 & & 0\\ & \ddots & \\ 0 & & 1\end{matrix} \\ \hline 0 \end{array}\right] \qquad g_i = \left[\begin{array}{c|c} 0 & \begin{matrix}1 & & 0\\ & \ddots & \\ 0 & & 1\end{matrix}\end{array}\right]$$

Let $\alpha : i \to j \in Q_1$, Then $\varphi_\alpha f_i = f_j \psi_\alpha$ and $g_j \varphi_\alpha = \chi_\alpha g_i \Rightarrow$ φ_α is rep. by $\varphi_\alpha = \left[\begin{array}{c|c} \psi_\alpha & \xi_\alpha \\ \hline 0 & \chi_\alpha \end{array}\right]$ w/ ξ_α is a $(\dim M_j - \dim N_j) \times (\dim M_i - \dim L_i)$ matrix. $\Rightarrow M \simeq L \oplus N \Leftrightarrow \xi_\alpha = 0 \ \forall \alpha \in Q_1$. If $M \not\simeq L \oplus N$ then $\xi_\alpha \neq 0$ for some α then for any $t \neq 0 \in K$

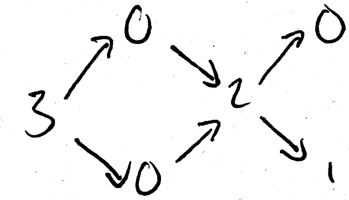
$$t \cdot \varphi_\alpha = \left[\begin{array}{c|c} \psi_\alpha & t\xi_\alpha \\ \hline 0 & \chi_\alpha \end{array}\right] \Rightarrow t \cdot M = (M_i, t \cdot \varphi_\alpha) \simeq M$$

Explicit isomoplism is given by $\left[\begin{array}{c|c} tI_{L_i} & 0 \\ \hline 0 & I_{N_i} \end{array}\right] \Rightarrow \dim \mathcal{O}_{L\oplus N} < \dim \mathcal{O}_M$.

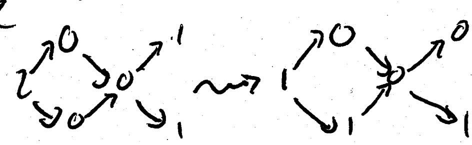
Ex. Let $Q: 1 \longrightarrow 2 \longleftarrow 3$. Recall Γ_Q:



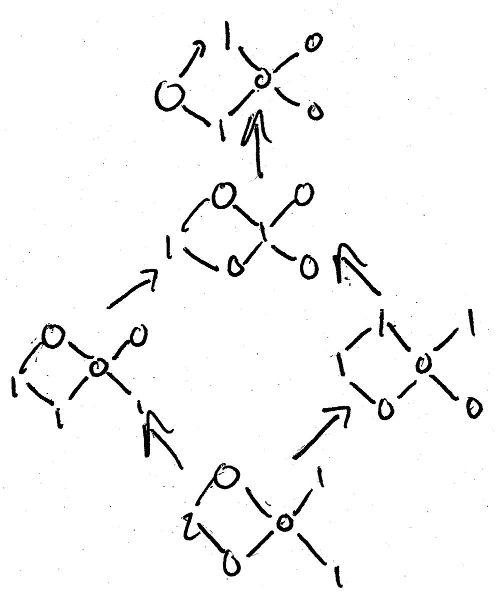
Fix $\underline{d} = (1,2,1)$, any $M \in E_{\underline{d}}$ is s.t. $M = \bigoplus c_i M_i$ w/ M_i indec. and $c_i \in \mathbb{Z}_{\geq 0}$ is its multiplicity. We can label the vertices of M_i by their multiplicities in Γ_Q, e.g.'s

$S(2) \oplus S(2) \oplus S(2) \oplus \begin{smallmatrix} 1\,3 \\ 2 \end{smallmatrix} \oplus \begin{smallmatrix} 1\,3 \\ 2 \end{smallmatrix} \oplus S(3) \leadsto$ 

Can use s.e. seq. to build reps. of orbits in $E_{\underline{d}}$

$\begin{smallmatrix} 1 \\ 2 \end{smallmatrix} \oplus S(2) \oplus S(3)$ comes from $0 \to S(2) \to \begin{smallmatrix} 1 \\ 2 \end{smallmatrix} \to S(1) \to 0$ and $\bigoplus d_i S(i)$ by replacing $S(2) \oplus S(1)$ by $\begin{smallmatrix} 1 \\ 2 \end{smallmatrix}$ 

since we reduce the multiplicity of $S(2)$ from $2 \to 1$ and $S(1)$ from $1 \to 0$ and increase mult. of $\begin{smallmatrix} 1 \\ 2 \end{smallmatrix}$ from $0 \to 1$. We construct a decomp. of $E_{\underline{d}}$ as follows:



$$\begin{tikzcd}
& \begin{smallmatrix} 1 \\ 2 \end{smallmatrix} \oplus \begin{smallmatrix} 3 \\ 2 \end{smallmatrix} & \\
& 2 \oplus \begin{smallmatrix} 1\,3 \\ 2 \end{smallmatrix} \arrow[u] & \\
\begin{smallmatrix} 1 \\ 2 \end{smallmatrix} \oplus 2 \oplus 3 \arrow[ur] & & \begin{smallmatrix} 3 \\ 2 \end{smallmatrix} \oplus 2 \oplus 1 \arrow[ul] \\
& 2 \oplus 2 \oplus 1 \oplus 3 \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

8.2 Quadratic Form of a Quiver

Let Q be quiver w/o orientated cycles, we define a quadratic form q to Q.

<u>Defn</u> The <u>quadratic form</u> $q(x)$ of a quiver Q is defined as:

$$q : \mathbb{Z}^n \to \mathbb{Z}, \quad q(x) = \sum_{i \in Q_0} x_i^2 - \sum_{\alpha \in Q_1} x_{s(\alpha)} x_{t(\alpha)}$$

(Here q has no dependency on α-orientation)

<u>Ex.</u> If $Q = 1 \longrightarrow 2 \longleftarrow 3$ then $q(x) = x_1^2 + x_2^2 + x_3^2 - x_1 x_2 - x_2 x_3$.

<u>Prop.</u> For any $M \in \operatorname{rep} Q$ w/ $\underline{\dim} M = \underline{d}$ we have $q(\underline{d}) = \dim \operatorname{Hom}(M,M) - \dim \operatorname{Ext}^1(M,M)$

<u>Pf.</u> Consider standard proj. resolution

$$0 \longrightarrow \bigoplus_{\alpha \in Q_1} d_{s(\alpha)} P(t(\alpha)) \xrightarrow{f} \bigoplus_{i \in Q_0} d_i P(i) \xrightarrow{g} M \longrightarrow 0.$$

Apply $\operatorname{Hom}(-,M)$ functor gives the exact seq.

$$0 \longrightarrow \operatorname{Hom}(M,M) \longrightarrow \bigoplus_{i \in Q_0} d_i \operatorname{Hom}(P(i),M) \longrightarrow \bigoplus_{\alpha \in Q_1} d_{s(\alpha)} \operatorname{Hom}(P(t(\alpha)),M) \longrightarrow \operatorname{Ext}^1(M,M) \longrightarrow 0$$

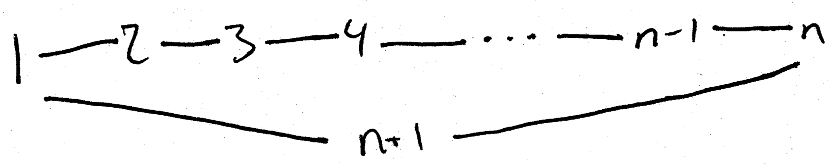
Since each $P(i)$ projective $\Rightarrow \dim \operatorname{Hom}(M,M) - \dim \operatorname{Ext}^1(M,M)$ equals:

$$\sum_{i \in Q_0} d_i \dim \operatorname{Hom}(P(i),M) - \sum_{\alpha \in Q_1} d_{s(\alpha)} \dim \operatorname{Hom}(P(t(\alpha)),M)$$

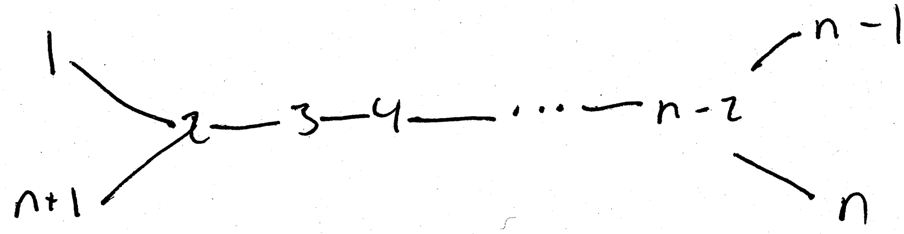
which in turn is equal to: $\displaystyle\sum_{i \in Q_0} d_i^2 - \sum_{\alpha \in Q_1} d_{s(\alpha)} d_{t(\alpha)}$. $\square$

Euclidean or Extended Dynkin Diagrams:

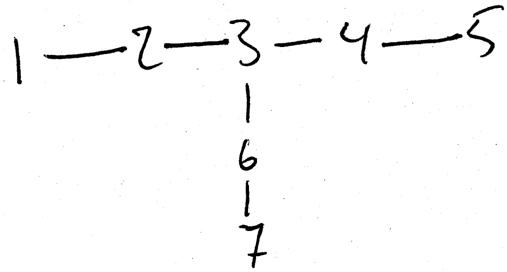
$\tilde{\mathbb{A}}_n$



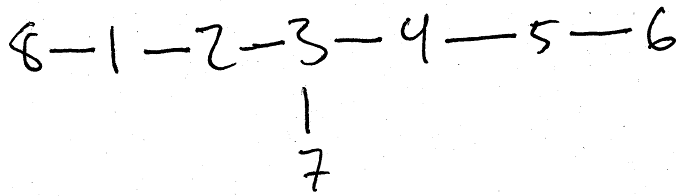
$\tilde{\mathbb{D}}_n$



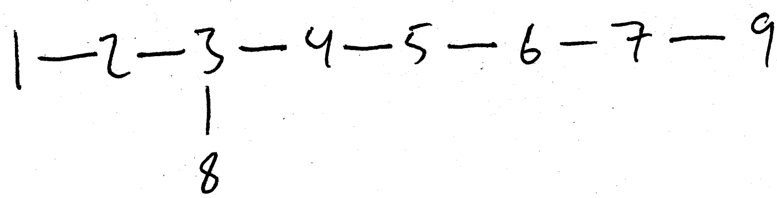
$\tilde{\mathbb{E}}_6$



$\tilde{\mathbb{E}}_7$



$\tilde{\mathbb{E}}_8$



Defn. Let q be a quadratic form.

1. q is positive definite if $q(x) > 0, \ \forall x \neq 0$
2. q is positive semi-definite if $q(x) \geq 0, \forall x \neq 0$

Lem. Assume that Q is connected. Let $\underline{d} = (d_i) \in \mathbb{Z}^n/\{0\}$ s.t. $\cancel{\gcd}(\underline{d}, x) = 0 \quad \forall\, x \in \mathbb{Z}^n$. Then.

(1) q is positive semi-definite

(2) $d_i \neq 0$, $\forall i$

(3) $q(x) = 0$ iff $x = \frac{a}{b}\underline{d}$ for some $a, b \in \mathbb{Z}$.

Pf. Let n_{ij} denote the number of arrows from $i \to j$ + number of arrows from $j \to i$ i.e, n_{ij} is the number of edges between i & j. We have

$$q(x) = \sum_{i=1}^{n} x_i^2 - \sum_{i=1}^{n} \sum_{j \mid j<i} n_{ij} x_i x_j \quad \text{and} \quad (x, y) = \sum_{i=1}^{n} 2x_i y_i - \sum_{i=1}^{n} \sum_{j \neq i} n_{ij} x_i y_j.$$

Sps $\underline{d}$ is as desired then $e_i = (0, \ldots, 1, \ldots, 0)$ is the ith standard basis of $\mathbb{Z}^n$, so $0 = (\underline{d}, e_i) = 2d_i - \sum_{j \mid j \neq i} n_{ij} d_j \Rightarrow d_i = \sum_{j \mid j<i} n_{ij} d_j$.

Since $n_{ij} \geq 0 \Rightarrow \exists\, i \in Q_0$ s.t. $d_i = 0 \Rightarrow$ for all neighbors j, $d_j = 0$

$\Rightarrow Q$ connected $\Rightarrow d_j = 0$, $\forall j \in Q_0$ contradiction $\Rightarrow$ (2) proved.

Let $\underline{x} \in \mathbb{Z}^n \Rightarrow \sum_{i=1}^{n} x_i^2 = \sum_i \frac{x_i^2}{d_i} \sum_{j \mid j<i} n_{ij} d_j$

$$\Rightarrow \sum_i x_i^2 = \sum_i \sum_{j \mid j<i} n_{ij} d_j \frac{x_i^2}{d_i} = \sum_i \sum_{j \neq i} \frac{n_{ij} d_j}{2} \frac{x_i^2}{d_i} = \sum_i \sum_{j \mid j<i} \left(\frac{n_{ij} d_j}{2} \frac{x_i^2}{d_i} + \frac{n_{ji} d_i}{2} \frac{x_j^2}{d_j} \right)$$

$$= \sum_i \sum_{j \mid j<i} \frac{n_{ij} d_i d_j}{2} \left(\frac{x_i^2}{d_i^2} + \frac{x_j^2}{d_j^2} \right) \Rightarrow q(x) = \sum_i \sum_{j \mid j<i} \frac{n_{ij} d_i d_j}{2} \left(\frac{x_i}{d_i} - \frac{x_j}{d_j} \right)^2$$

Thus $q(x) \geq 0$ since $d_i, d_j > 0$ & $n_{ij} \geq 0 \Rightarrow$ (1) proved.

$q(x) = 0$ iff $\frac{x_i}{d_i} = \frac{x_j}{d_j}$ for $n_{ij} \neq 0 \Rightarrow Q$ connected that proves (3). $\square$

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<u>Thm.</u> Let Q be a connected quiver. Then,

1) q is positive definite iff Q is of Dynkin type $\mathbb{A}_n, \mathbb{D}_n, \mathbb{E}_{6,7,8}$

2) q is positive semi-definite iff Q is Euclidean $\tilde{\mathbb{A}}_n, \tilde{\mathbb{D}}_n, \tilde{\mathbb{E}}_{6,7,8}$ or Dynkin $\mathbb{A}_n, \mathbb{D}_n, \mathbb{E}_{6,7,8}$.

<u>Pf.</u>

We first show q is pos. semi-definite if Q is Euclidean. It suffices to find a vector δ for each previous Euclidean diagram.

$\tilde{\mathbb{A}}_n \Rightarrow \delta = (1,1,1,1,\dots,1,1,1)$

$\tilde{\mathbb{D}}_n \Rightarrow \delta = (1,2,2,2,\dots,2,1,1,1)$

$\tilde{\mathbb{E}}_6 \Rightarrow \delta = (1,2,3,2,1,2,1)$

$\tilde{\mathbb{E}}_7 \Rightarrow \delta = (2,3,4,3,2,1,2,1)$

$\tilde{\mathbb{E}}_8 \Rightarrow \delta = (2,4,6,5,4,3,2,3,1)$

Easy to check that δ satisfies $(\delta, \underline{x}) = 0 \quad \forall x \neq 0$.

Conversely, sps q pos-semi-definite. & Q not Euclidean or Dynkin.

$\Rightarrow \exists Q' \subset Q$ s.t. Q' Euclidean, let q' be quad form of Q' and δ the relevant dim. vector.

If $Q_0 = Q_0' \Rightarrow Q_1' \subset Q_1 \Rightarrow 0 = q'(\delta) > q(\delta)$ ⊗

If $Q_0' \subset Q_0'$ chose $i_0 \in Q_0$ connected to $j_0 \in Q_0^2$

Define $\underline{x}$ s.t. $x_i = 2\delta_i$, $\forall i \in Q_0'$, $x_{i_0} = 1$, and $x_j = 0$ for $j \notin Q_0$

$\Rightarrow q(\underline{x}) \geq q'(2\delta) + 1 - 2\delta_{j_0} = 1 - 2\delta_{j_0} < 0$ a contradiction ⊗

$\Rightarrow q$ pos. semi-definite $\Rightarrow Q$ is Euclidean or Dynkin.

Next, if q is pos. definite $\Rightarrow$ Q is Dynkin since $q(x)=0$ iff Q is Euclidean.

NTS Q of Dynkin $\mathbb{A}_n, \mathbb{D}_n, \mathbb{E}_{6,7,8} \Rightarrow q$ pos. definite.

Sps Q as desired, let $\overline{Q}$ be the Euclidean diagram extending Q to $n+1$ vertices, and $\overline{q}$ the corresponding quad. form.

Sps $\exists\, \underline{x} \in \mathbb{Z}^n \setminus \{0\}$ s.t. $q(\underline{x}) \leq 0$, let $\overline{x} \in \mathbb{Z}^{n+1}$ be define as

$$\overline{x}_i = \begin{cases} x_i & i \neq n+1 \\ 0 & i = n+1 \end{cases}$$

$\Rightarrow \overline{q}(\overline{x}) = q(\underline{x}) \leq 0 \Rightarrow \overline{q}(\overline{x}) = 0$ since $\overline{q}$ pos. semi-definite.

$\Rightarrow \overline{x} = \frac{a}{b}\delta$ for some $a, b \in \mathbb{Z}$ but this is a contradiction since $\overline{x}_{n+1} = 0 \Rightarrow q$ must be positive definite. $\square$

8.3 Roots.

Let $\underline{x} \in \mathbb{Z}^n \setminus \{0\}$ then $\underline{x}$ is a real root if $q(\underline{x}) = 1$ and $\underline{x}$ is an imaginary root if $q(\underline{x}) = 0$.

It follows that every root of q is of the form $a_1 e_1 + \dots + a_n e_n \Leftrightarrow (a_1, \dots, a_n)$ w/ $a_i \in \mathbb{Z}$. If $\alpha = (a_1, \dots, a_n)$ is a root then α is a positive root if $a_i \geq 0\ \forall i$. Similarly α is negative root if $a_i \leq 0\ \forall i$.

Φ is the set of all roots. Φ_+ set of all pos. roots. Φ_- set of all neg. roots.

Lem.

1) e_i is a real root $\forall i \in Q_0$

2) If α is a root then $-\alpha$ is a root

3) If α is a root of Q Euclidean and α is not the imaginary root δ, as previous, then $\alpha - \delta$ is a root.

4) If q is pos. semi-definite then each root is either pos. or neg. and $\Phi = \Phi_- \cup \Phi_+$ and $\Phi_- = \Phi_+$.

Pf. 1) follows from defn of q.

2) holds since $q(-\alpha) = q((-1)\alpha) = (-1)^2 q(\alpha) = q(\alpha)$.

3) $q(\alpha - \delta) = q(\alpha) + q(-\delta) + (\alpha, -\delta) \Rightarrow q(\alpha - \delta) = q(\alpha)$.
Since $q(-\delta) = (\alpha, \delta) = 0$.

$\square$

4) Omitted.

Cor. If Q is of Dynkin type, then there are finitely many roots and each root is a real root.

Pf. There are no imaginary roots since q pos. definite. Let α be a root of q. Extend Q to $\bar{Q}$ Euclidean. w/ new vertex i_0 and $\bar{q}$ the form $\bar{Q}$. w/ δ the imaginary root of $\bar{Q}$.
$\Rightarrow \alpha - \delta$ is a root, negative at vertex $i_0 \Rightarrow$ negative root.

It follows that $\forall i \in Q_0$, $\alpha_i \le \delta_i \Rightarrow$ finite possibilities of α. $\square$

We can then list all positive roots of corresponding Dynkin types via direct computation. These can be found in the text.

8.4 Gabriel's Theorem

Relate the dimension of the orbit to the quadratic form.

<u>Prop.</u> Let Q be a connected quiver, $M \in \operatorname{rep} Q$, $\underline{\dim} M = \underline{d}$. Then,

$$\operatorname{codim} \mathcal{O}_M = \dim \operatorname{End}(M) - q(\underline{d}) = \dim \operatorname{Ext}^1(M, M).$$

<u>Pf.</u> We have $\dim \mathcal{O}_M = \dim G_{\underline{d}} - \dim \operatorname{Aut}(M)$

$\operatorname{Aut}(M)$ is open subgroup of $\operatorname{End}(M) \Rightarrow \dim \operatorname{Aut}(M) = \dim \operatorname{End}(M)$

$\dim GL_{d_i} = d_i^2 \Rightarrow \dim G_{\underline{d}} = \sum_{i \in Q_0} d_i^2$. Thus,

$$\begin{aligned} \operatorname{codim} \mathcal{O}_M &= \dim E_{\underline{d}} - \dim \mathcal{O}_M \\ &= \underbrace{\sum_{\alpha \in Q_1} d_{s(\alpha)} d_{t(\alpha)} - \sum_{i \in Q_0} d_i^2}_{= -q(\underline{d})} + \dim \operatorname{End}(M) \end{aligned}$$

The second eq. follows from previous result. □

<u>Cor.</u> If $q(\underline{d}) \leq 0$ then there are infinitely many isoclasses of representations of Q w/ dimension vector $\underline{d}$.

<u>Pf.</u> Let $\underline{d}$ s.t. $q(\underline{d})$ and $M \in \operatorname{rep} Q$ s.t. $\underline{\dim} M = \underline{d}$

Then $\operatorname{codim} \mathcal{O}_M \geq \dim \operatorname{End}(M) \geq 1 \Rightarrow$ dimension of $E_{\underline{d}}$ is <u>strictly</u> greater than the dimension of any orbit $\mathcal{O}_M$.

This implies the number of orbits is infinite. □

Gabriel's Theorem. Let Q be a connected quiver. Then,

1) Q is of finite representation type iff Q is of Dynkin type $\mathbb{A}_n, \mathbb{D}_n,$ or $\mathbb{E}_{6,7,8}$

2) If Q is of Dynkin type $\mathbb{A}_n, \mathbb{D}_n, \mathbb{E}_{6,7,8}$ then the dimension vector induces a bijection ψ from isoclasses of indec. reps. of Q to the set of positive roots Φ_+:

$$\psi : \operatorname{ind} Q \longrightarrow \Phi_+ \quad \text{s.t.} \quad \psi : M \longrightarrow \underline{\dim}\, M$$

Pf. We prove 2) then 1).

To prove 2) we first show ψ is welltodefined.

Let M be an indecomposable ~~notate~~ representation, NTS $q(\underline{\dim}\, M) = 1$, suffices to show $\operatorname{End} M \simeq k$ and $\dim \operatorname{Ext}^1(M,M) = 0$.

To show $\operatorname{End} M \simeq k$ we proceed by induction on $\dim M$.

↳ M simple follows immediately

↳ M has dim. > 1 and $\operatorname{End} L \simeq k$ for $L \subset M$, L indec.

Sps ~~M~~ $\operatorname{End} M \not\simeq k$, M indec $\Rightarrow$ every $f \in \operatorname{End} M$ is of the form $f = \lambda 1_M + g$, g nilpotent $\in \operatorname{End} M$

$\Rightarrow g^m = 0$ for $m \geq 2$, say $m = 2$, choose g s.t. $\dim(\operatorname{im} g)$ is minimal

$g^2 = 0 \Rightarrow \operatorname{im} g \subset \ker g \Rightarrow \exists L$ indec $\subseteq \ker g$ s.t.

$\operatorname{im} g \cap L \neq \{0\}$

Pf. of Gabriels cont.

Let $\pi : \ker g \to L$ be the projection

i the nonzero morphism given by $\text{incl} : \operatorname{im} g \to \ker g$ & π.

$$\begin{tikzcd} \operatorname{im} g \arrow[r, "\text{incl.}"] \arrow[rr, bend right, "i"'] & \ker g \arrow[r, "\pi"] & L \end{tikzcd}$$

This implies the composition

$$M \xrightarrow{g} \operatorname{im} g \xrightarrow{i} L \xrightarrow{\text{incl}} M$$

is a nonzero endomorphism w/ square $= 0$,

image $= i(\operatorname{im} g)$. g minimal $\Rightarrow \dim i(\operatorname{im} g) \geq \dim \operatorname{im} g$

$\Rightarrow i$ injective. So the short exact sequence

$$0 \to \operatorname{im} g \xrightarrow{i} L \to \operatorname{coker} i \to 0$$

which we apply $\operatorname{Hom}(-, L)$ functor gives surj. morphism

$$\operatorname{Ext}^1(L, L) \to \operatorname{Ext}^1(\operatorname{im} g, L) \to 0 \qquad (\ast)$$

By induction $\dim \operatorname{Hom}(L, L) = 1$, q pos. definite $\Rightarrow \dim \operatorname{Ext}^1(L, L) = 0$

So $(\ast)$ shows $\operatorname{Ext}^1(\operatorname{im} g, L) = 0$.

Pf. of Gabriels cont.

Consider the diagram (commutative) w/ exact rows w/ bottom rows as a push out of the top row along morphism π.

$$\begin{tikzcd}
0 \arrow[r] & \ker g \arrow[r, "u"] \arrow[d, "\pi"] & M \arrow[r, "g"] \arrow[d, "j_2"] & \operatorname{im} g \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & L \arrow[r, "j_1"'] & X \arrow[r] & \operatorname{im} g \arrow[r] & 0
\end{tikzcd}$$

$\operatorname{Ext}^1(\operatorname{im} g, L) = 0 \Rightarrow$ bottom row splits so

$\exists\, h: X \to L$ s.t. $hj_1 = 1_L$. Let $v: L \to \ker g$

be the inclusion of direct summand. so $\pi v = 1_L$

Then we construct $hj_2: M \to L$ and $uv: L \to M$

s.t. $hj_2 uv = hj_1 \pi v = 1_L 1_L = 1_L \Rightarrow L$ is direct summand

of M. But M indec $\Rightarrow$ $L = 0$ or $L = M$.

But $L \neq 0$ since $\operatorname{im} g \cap L \neq \{0\}$ $\Big\}$ $\Rightarrow$ Contradiction.

And $L \neq M$ since $L \subset \ker g$, $g \neq 0$

So $\dim \operatorname{End} M = 1$, q pos. definite $\Rightarrow \dim \operatorname{Ext}^1(M, M) = 0$

and $q(\underline{\dim} M) = 1$. Hence $\underline{\dim} M$ is a pos. root

ψ is well-defined.

Pf. of Gabriel's thm.

ψ injective: Let $M, M' \in \text{rep}\, Q$ indec s.t. $\underline{\dim}\, M = \underline{\dim}\, M'$

in Dynkin type, indec. reps have no self-extensions.
$\Rightarrow \mathcal{O}_M$ & $\mathcal{O}_{M'}$ both have codim $= 0 \Rightarrow M \cong M' \Rightarrow \psi$ inj.

ψ surj.: Let Q be Dynkin, $\underline{d}$ a pos root. $M \in \text{rep}\, Q$ s.t. $\underline{\dim}\, M = \underline{d}$ and $\mathcal{O}_M$ of max dim. in $E_{\underline{d}}$. NTS M indec.

Sps $M = M_1 \oplus M_2$, we first show $\text{Ext}^1(M_1, M_2) = \text{Ext}^1(M_2, M_1) = 0$
Sps $\text{Ext}^1(M_1, M_2) \neq 0 \Rightarrow \exists$ non-split short exact seq. of the form

$$0 \longrightarrow M_2 \longrightarrow E \longrightarrow M_1 \longrightarrow 0$$

Here $\underline{\dim}\, E = \underline{\dim}\, M$. Then previous result $\Rightarrow \dim \mathcal{O}_M < \dim \mathcal{O}_E$ Contradicting maximality of $\mathcal{O}_M$. Thus $\text{Ext}^1(M_1, M_2) = 0$ & by symmetry $\Rightarrow \text{Ext}^1(M_1, M_2) = \text{Ext}^1(M_2, M_1) = 0$.

Then, $1 = q(\underline{d}) = \dim \text{Hom}(M_1 \oplus M_2, M_1 \oplus M_2) - 0 \geq 2$

A contradiction, thus M indecomposable. And $\psi(M) = \underline{d}$

Thus ψ is surjective.

Therefore, ψ is a bijection. So 2) is complete.

Pf. of Gabriel's cont.

We now prove 1).

Sps Q is not Dynkin, then $\exists\ \underline{d} \neq 0$ s.t. $q(\underline{d}) \leq 0$

then there are infinitely many isoclasses of reps. w/ dim. vector $\underline{d}$.

Each rep. is a finite direct sum of indec. reps.

Therefore the number of isoclasses of indec. reps. is infinite.

This shows 1) and thus we have proven Gabriel's Theorem $\square$