THE KNITTING ALGORITHM FOR $\mathbb{AD}$ TYPE QUIVERS

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Abstract. We present an expository note on the theory of quiver representations along with a description of the knitting algorithm, which is used to compute the Auslander-Reiten quiver Γ_Q for $\mathbb{AD}$ type quivers. We focus on constructing Γ_Q utilizing the Coxeter matrix Φ, highlighting the benefit of this method with regards to direct computation.

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1. Introduction

A quiver is a directed graph, defined using a tuple of the form $Q = (Q_0, Q_1, s, t)$ where Q_0 is a set of *vertices*, Q_1 a set of *arrows* between vertices, and $s, t : Q_1 \to Q_0$ are mappings which map arrows to their *source* and *target* vertex respectively. First introduced in [Gab72], quivers have integrated themselves into various research fields across mathematics. From areas in complex geometry and mathematical physics [CG97, Soi19, Gin09] to more combinatorial flavored topics [DW11, FM$^+$17] and recently even machine learning [GW22, AJ21].

Introduced in [ARS97], Auslander-Reiten theory studies the representation of Artinian rings amongst other things. One tool utilized in this theory is the Auslander-Reiten quiver, denoted as Γ_Q, a map which allows the contemporary mathematician to traverse the category of representations of a quiver rep Q. The goal of this paper is to explain the knitting algorithm, focusing on the construction of Γ_Q utilizing the Coxeter functor and matrix Φ.

Fortunately quivers, as they are presented in this paper, are relatively easy to understand. Requiring only a familiarity with linear algebra. Further material on quiver representations and Auslander-Reiten theory can be found in [Bri08, Uni12,

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DW17, DW05], we highlight [Sch14] as much of the material in this paper was derived from it.

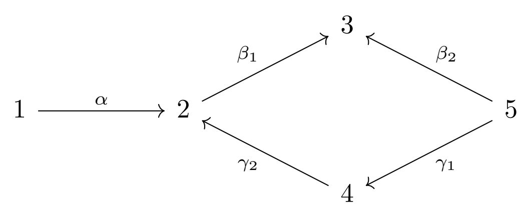
In this paper we take k to be an arbitrary algebraically closed field, for intuitive purposes it often suffices to think of k as $\mathbb{C}$. For a quiver Q, we label the vertices with positive integers and the arrows with greek letters.

In section 2 we introduce quiver representations and some initial results providing examples to illustrate the ideas more readily, also included is a brief detour to categories and finite-dimensional algebras providing further context into what motivates quiver theory in subsection 2.2. Section 3 introduces the Auslander-Reiten quiver Γ_Q formally, and illustrates how one can traverse it. Lastly, section 4 discusses the aforementioned knitting algorithm.

2. Quiver Representations

We begin with an example of an elementary quiver, before remarking on some properties of quivers setting the stage for the rest of this section.

Example 2.1. Let $Q = (Q_0, Q_1, s, t)$ be the quiver as shown below.



It follows that $Q_0 = \{1, 2, 3, 4, 5\}$, $Q_1 = \{\alpha, \beta_1, \beta_2, \gamma_1, \gamma_2\}$, and $s(\gamma_1) = 5$ and $t(\gamma_1) = 4$.

Quivers can be quite expressive, we allow for (self) loops and multiple arrows between the same two vertices. If Q_0 and Q_1 are both finite sets, then we say that Q itself is finite. For this paper we assume all quivers discussed are finite. Other ideas from graph theory (connected, cycle, paths) apply to quivers as well, for a primer on graph theory we recommend [Har11].

2.1. **Representations of Quivers.**

The broad goal of representation theory is to take an algebraic structure and use linear algebra to gain a new perspective of sorts. The motivation for this arises since linear algebra is a well-studied and understood field. By using representation theoretic techniques we can gain, at the very least, a new perspective on the particular problem at hand, often resulting in a stronger understanding.

Remark. A large branch of representation theory is the representation theory of finite groups, in this subject a representation is a group homomorphism of the form $\rho : G \to V$ where $V = \mathrm{GL}_m(k)$ is an m-dimensional k-vector space. Quivers hold use in this subject by means of the *McKay quiver*, a quiver which encodes the data of a group representation in its structure, introduced in [McK80].

For the representations of quivers, the idea is quite simple, vertices become vector spaces and arrows become linear maps. Formally, we define a representation of a quiver as follows.

Definition 2.2. Let $Q = (Q_0, Q_1, s, t)$ be a quiver, a *representation*

$$M = (M_i, \varphi_\alpha)_{i \in Q_0, \alpha \in Q_1}$$

is a collection of k-vector spaces M_i one for each vertex, and linear maps

$$\varphi_\alpha : M_{s(\alpha)} \to M_{t(\alpha)}$$

one for each arrow.

A representation M is finite-dimensional if each vector space M_i is also finite-dimensional. In this case we define the *dimension vector* of M to be $\underline{\dim} M = (\dim M_i)_{i \in Q_0}$ an n-tuple of the dimensions of each vector space in M where n is the number of vertices in Q. An element of M is another n-tuple defined as $(m_i)_{i \in Q_0}$ with $m_i \in M_i$.

Definition 2.3. Let Q be a quiver with two representations $M = (M_i, \varphi_\alpha)$ and $N = (N_i, \psi_\alpha)$. A *morphism* of representations $f : M \to N$ is a collection of linear maps $f = (f_i)$, $f_i : M_i \to N_i$ such that for each arrow $i \xrightarrow{\alpha} j \in Q_1$ the following diagram commutes.

$$\begin{array}{ccc} M_i & \xrightarrow{\varphi_\alpha} & M_j \\ {\scriptstyle f_i}\downarrow & & \downarrow{\scriptstyle f_j} \\ M'_i & \xrightarrow{\psi_\alpha} & M'_j \end{array}$$

That is, $f_j \circ \varphi_\alpha(m) = \psi_\alpha \circ f_i(m)$ for all $m \in M_i$.

We now introduce direct sums.

Definition 2.4. Let Q be a quiver with representations $M = (M_i, \varphi_\alpha)$ and $N = (N_i, \psi_\alpha)$. Then,

$$M \oplus N = \left(M_i \oplus N_i, \left[\begin{array}{c|c} \varphi_\alpha & 0 \\ \hline 0 & \psi_\alpha \end{array} \right] \right)_{i \in Q_0, \alpha \in Q_1}$$

is another representation of Q called the *direct sum* of M and N.

If $M \neq 0$ is a representation of Q, then M is said to be *indecomposable* if M cannot be written as the direct sum of two non-zero representations. Indecomposable representations are analogous to primes to the integers, acting as a building block for all other representations.

The following example, adapted from [Sch14, Example 1.7], illustrates the three previous concepts.

Example 2.5. Suppose we have Q as in **??**. We first define two representations of Q as follows;

$$L\colon \qquad k^2 \xrightarrow{\ T\ } k^2 \xleftarrow{\ S\ } k$$

$$M\colon \qquad k \xrightarrow{\ 1\ } k \xleftarrow{\ 0\ } 0$$

with maps T and S given as

$$T = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = 2I \qquad S = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

We can now consider a morphism of representations $f : L \to M$, illustrated below as dashed arrows.

$$\begin{array}{ccccccc} L & \qquad & k^2 & \xrightarrow{T} & k^2 & \xleftarrow{S} & k \\ \Big\downarrow{\scriptstyle f} & & \Big\downarrow{\scriptstyle f_1} & & \Big\downarrow{\scriptstyle f_2} & & \Big\downarrow{\scriptstyle f_3} \\ M & & k & \xrightarrow{1} & k & \xleftarrow{0} & 0 \end{array}$$

It follows then, that $f_1 : k^2 \to k$ is of the form $\begin{bmatrix} a & b \end{bmatrix}$, likewise $f_2 : k^2 \to k$ has form $\begin{bmatrix} c & d \end{bmatrix}$, and lastly $f_3 : k \to 0$ is just the zero map. To preserve commutativity we then get the relations $a + b = c + d = 0$ and thus $b = -a$ and $d = -c$. However we also have the added condition that $a = 2c$, this follows from the need to have the diagram commute, as $T = 2I$ and we need to preserve the commutativity. Since one choice of scalar $a \in k$ completely determines the morphism we see that $\mathrm{Hom}(L, M) \cong k$.

Now consider a new representation $N \cong L \oplus M$, first we compute $L \oplus M$

$$L \oplus M : \qquad k^2 \oplus k \xrightarrow{\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}} k^2 \oplus k \xleftarrow{\begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 0 \end{bmatrix}} k \oplus 0 \ .$$

Thus, we see that N is the representation

$$N : \qquad k^3 \xrightarrow{\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}} k^3 \xleftarrow{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}} k \ .$$

A broad goal of representation theory (of quivers) is to classify all representations of a quiver Q and morphisms between them up to isomorphism. The well-known Krull-Schmidt theorem shows that it suffices to classify the *indecomposable* representations and the morphisms between them. We list the theorem here, but omit the proof. Proving existence is straightforward but uniqueness is less so, requiring a series of prior results, we reference [Hun12] for a complete proof.

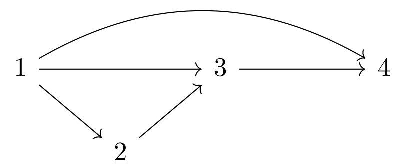
Theorem 2.1 (Krull-Schmidt)**.** *Let Q be a quiver and M a representation of Q, then*

$$M \cong M_1 \oplus M_2 \oplus \cdots \oplus M_t$$

where the M_i are indecomposable representations of Q that are unique up to order.

The last notion on representations we need to cover are projective, injective, and simple representations. Their use will come up in the later sections, but we will define them here. Formally, we have to define the dimensions of each vector space in the collection and then define the linear maps. In our case, we focus strictly on the dimension of each vector space, omitting the details of the maps as they are not necessarily relevant towards the Auslander-Reiten quiver. Let Q be the following quiver.

(2.1)



A *simple representation* $\mathcal{S}(i)$ of Q is determined by having each vector space in its collection be 0 except for the ith vertex, which is of one dimension. That is,

$$\mathcal{S}(i)_j := \begin{cases} k & j = i \\ 0 & j \neq i \end{cases}.$$

It follows then that the dimension vector of $\mathcal{S}(i)$ is equal to 0 everywhere and 1 at the ith index, e.g. if Q is as above then $\mathcal{S}(3) \in \operatorname{rep} Q$ is such that $\underline{\dim}\mathcal{S}(3) = (0, 0, 1, 0)$.

A *projective representation* $\mathcal{P}(i)$ of Q is determined by the number of paths (see Definition 2.8) from vertices i to j, for all vertices j. It may be easier to consider a working example, let $\mathcal{P}(1) \in \operatorname{rep} Q$. Then the dimension of $\mathcal{P}(1)_1$ is just 1 as there is just the trivial path going from vertex $1 \to 1$. The dimension of $\mathcal{P}(1)_2$ is likewise just 1 as there is only one path from $1 \to 2$. The dimension of $\mathcal{P}(1)_3$ is 2 since there are two paths from $1 \to 3$; the first being $1 \to 3$ and the second being $1 \to 2 \to 3$. The dimension of $\mathcal{P}(1)_4$ is 3 since there are three paths from $1 \to 4$; the first $1 \to 4$, second $1 \to 3 \to 4$, and last $1 \to 2 \to 3 \to 4$. We see then that $\underline{\dim}\mathcal{P}(1) = (1, 1, 2, 3)$.

An *injective representation* $\mathcal{I}(i)$ of Q is similar to projective representations, but instead determined by the number of paths j to i, for all vertices j. We consider $\mathcal{I}(4) \in \operatorname{rep} Q$. The dimension of $\mathcal{I}(4)_1$ is 3 since there are three paths from $1 \to 4$, the ones described above for $\mathcal{P}(1)_4$. The dimension of $\mathcal{I}(4)_2$ is 1 since there is just the path $2 \to 3 \to 4$. Likewise, the dimensions of $\mathcal{I}(4)_3$ and $\mathcal{I}(4)_4$ are both just 1. So $\underline{\dim}\mathcal{I}(4) = (3, 1, 1, 1)$.

Remark. If $\mathcal{S}(i) = \mathcal{P}(i)$ (resp. $\mathcal{S}(i) = \mathcal{I}(i)$) then Q has a *sink* (resp. *source*) at vertex i.

We list all $\mathcal{S}$, $\mathcal{P}$, and $\mathcal{I}$ representations in Figure 1. Moreover, each $\mathcal{S}(i)$, $\mathcal{P}(i)$, or $\mathcal{I}(i)$ are all indecomposable.

We further note that it suffices to describe representations in terms of its corresponding dimension vector, this allows us to quickly present a representation without having to draw it graphically or explicitly define its linear maps.

2.2. **The Category** $\operatorname{rep} Q$**.** Without detailing too much ring theory we recall a few foundational definitions.

Definition 2.6. Let A be a k-vector space equipped with an additional binary operation $\cdot : A \times A \to A$. Then A is a *k-algebra* if for all $a, b, c \in A$ and $\lambda, \mu \in k$:

(1) $(a + b) \cdot c = a \cdot c + b \cdot c$,
(2) $c \cdot (a + b) = c \cdot a + c \cdot b$,
(3) $(\lambda a) \cdot (\mu b) = (\lambda\mu)(a \cdot b)$.

Alternatively, one can think of A as a ring with unity such that A has a k-vector space structure.

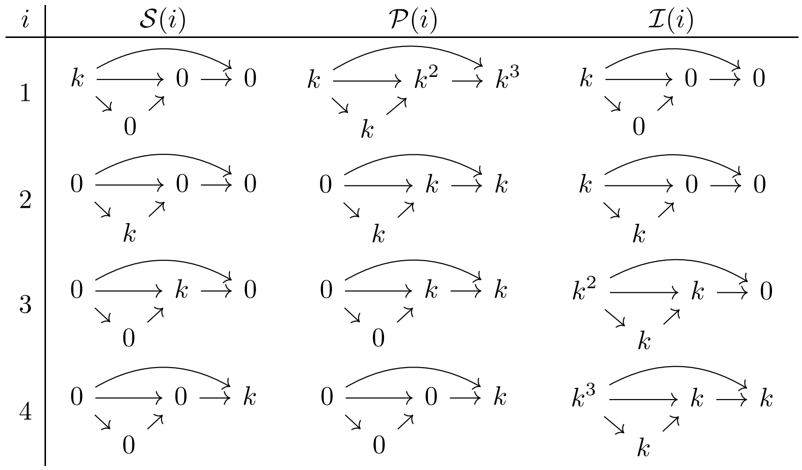


FIGURE 1. $\mathcal{S}$, $\mathcal{P}$, and $\mathcal{I}$ representations for Q as in (2.1).

Definition 2.7. Let R be a ring with unity, a (right) R-module M is an abelian group equipped with a binary operation called a (right) R-action

$$M \times R \to M \qquad (m, r) \mapsto mr$$

such that for all $m, n \in M$ and $r, s \in R$:

(1) $(m + n)r = mr + nr$,
(2) $m(r + s) = mr + ms$,
(3) $m(rs) = (mr)s$,
(4) $m1 = m$.

Fix a quiver Q, the set of all representations of Q over our fixed field k is denoted as $\operatorname{rep} Q$. Informally, a category is a collection of objects and morphisms between objects such that two properties are satisfied; the first being associativity, and the second being identity. In the case of $\operatorname{rep} Q$ our objects are representations of Q and the morphisms between objects are morphisms between representations.

To give further context into why quivers are a valuable field of study we must define the *path algebra* of Q. We first define the natural notion of a path on Q and then the path algebra kQ.

Definition 2.8. Let $Q = (Q_0, Q_1, s, t)$ be a quiver with $i, j \in Q_0$. A *path* c from i to j with length l in Q is a sequence

$$c = (i \mid \alpha_1, \alpha_2, \ldots, \alpha_l \mid j)$$

where $\alpha_h \in Q_1$ such that $s(\alpha_1) = i$, $s(\alpha_h) = t(\alpha_{h-1})$ for $2 \leq h \leq l$, and $t(\alpha_l) = j$.

Definition 2.9. Let Q be a quiver, the *path algebra* kQ of Q is a k-algebra whose basis is the set of all paths in Q and multiplication is defined on two basis elements c and d as follows

$$cd = \begin{cases} c \cdot d & s(d) = t(s) \\ 0 & s(d) \neq t(s) \end{cases}$$

Where $c \cdot d$ is the concatenation of paths, i.e. if $c = (i \mid \{\alpha\} \mid j)$ and $d = (j \mid \{\beta\} \mid k)$ then $c \cdot d = (i \mid \{\alpha\}, \{\beta\} \mid k)$.

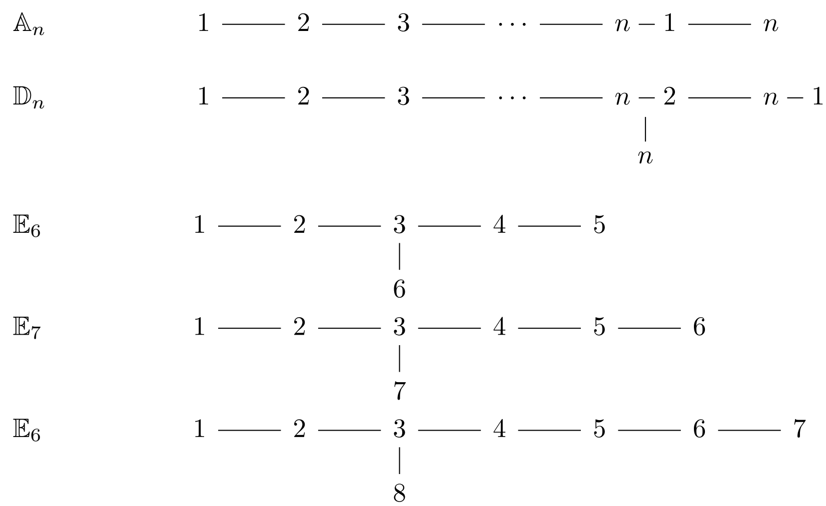


FIGURE 2. Dynkin diagrams $\mathbb{A}_n$, $\mathbb{D}_n$ and $\mathbb{E}_{6,7,8}$.

This then sets us up for the following theorem.

Theorem 2.2. *The category of finite-dimensional representations,* $\operatorname{rep} Q$*, is equivalent to the category of finite-dimensional (right) kQ-modules,* $\operatorname{mod} kQ$*.*

The proof of this is quite straightforward, but does require a somewhat intimate understanding of category theory. The idea is to construct two functors F and G and prove that their composition is the desired identity. An often cited proof can be found in [CB92].

This result, is often paired with the well-celebrated Gabriel's theorem. We first note that an underlying Dynkin diagram of a quiver Q, denoted Δ_Q, is achieved by removing arrows from Q and replacing them with edges, i.e. removing any orientation of arrows. We list the relevant Dynkin diagrams in Figure 2.

Theorem 2.3 (Gabriel 1972)**.** *Let Q be a connected quiver.*

(1) Then Q is of finite representation type if and only if the underlying Dynkin diagram, Δ_Q, of Q is of Dynkin type $\mathbb{A}$, $\mathbb{D}$, or $\mathbb{E}$.
(2) If Q has underlying Dynkin type $\mathbb{A}$, $\mathbb{D}$, or $\mathbb{E}$ then the dimension vector induces a bijection from isoclasses of indecomposable representations of Q to the set of positive roots Φ (of the relevant quadratic form):

$$\Psi : \operatorname{ind} Q \to \Phi_+ \qquad \Psi : M \mapsto \underline{\dim} M.$$

The implications of Gabriel's theorem are quite profound, it bridges the study of finite-dimensional algebras to that of quivers. Furthermore, the set of indecomposable modules of kQ are in a one-to-one correspondence with the set of indecomposable representations of Q, we explicitly note that simple representations map to simple modules and the same applies towards projective and injective representations. Gabriel's theorem sparked much interest in quivers and has led to the development of the field as a whole. It directly led to Kac's theorem [Kac82], which extended itself to the study of Lie algebras. Once again, we omit a proof here as it is quite lengthy and requires some more background on algebraic varieties and quadratic forms. There are many expository notes which lead up to the proof of Gabriel's theorem [Bri08, Cum11, Len19, Hal21, Aku22].

3. The Auslander-Reiten Quiver Γ_Q

As we have seen, by the Krull-Schmidt theorem, the goal of representation theory is to classify all indecomposable representations and the morphisms between them. The so-called *Auslander-Reiten* quiver Γ_Q gives an approximation of $\operatorname{rep} Q$, and in the case of Q being Dynkin type $\mathbb{A}$, $\mathbb{D}$, or $\mathbb{E}$ (that is when Q has finite representation type) provides complete information about $\operatorname{rep} Q$. Broadly, Γ_Q encodes three aspects of $\operatorname{rep} Q$; indecomposable representations, "irreducible" morphisms, and exact sequences. In this paper we omit any detailed mention of exact sequences (and *almost split sequences*, see [AR75]) as they require more background in homological algebra. For further detail on Auslander-Reiten quivers see [Zel05]. Lastly, for the duration of this paper we assume that Q is either Dynkin type $\mathbb{A}$ or $\mathbb{D}$, unless otherwise stated.

For a fixed quiver Q, Γ_Q is a quiver whose vertices are the indecomposable representations in $\operatorname{rep} Q$ and the arrows between vertices are the irreducible morphisms (informally, these are morphisms between indecomposable representations which cannot be factored through another representation). We present the following adapted from [Sch14, Example 1.14] and [Zel05, Example 1.3].

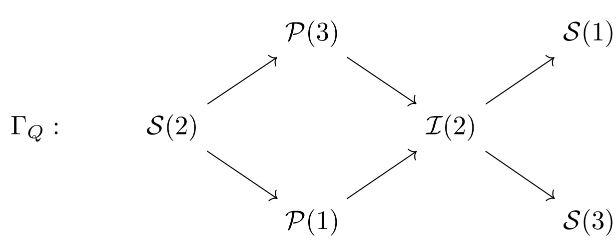
Example 3.1. Suppose we have the same elementary quiver Q as in Example 2.5. Note that $\Delta_Q = \mathbb{A}_3$, and thus Γ_Q will provide full information about $\operatorname{rep} Q$. One can directly compute the isoclasses of indecomposable representations in $\operatorname{rep} Q$ using the technique discussed in Section 2. We see that there are precisely six such representations;

$$\mathcal{S}(1): \ k \longrightarrow 0 \longleftarrow 0 \qquad \mathcal{S}(2): \ 0 \longrightarrow k \longleftarrow 0$$

$$\mathcal{S}(3): \ 0 \longrightarrow 0 \longleftarrow k \qquad \mathcal{I}(2): \ k \longrightarrow k \longleftarrow k$$

$$\mathcal{P}(1): \ k \longrightarrow k \longleftarrow 0 \qquad \mathcal{P}(3): \ 0 \longrightarrow k \longleftarrow k$$

The Auslander-Reiten quiver is of the form:



For representations L, M, and N in Example 2.5 we get the decompositions;

$$L \cong \mathcal{P}(1) \oplus \mathcal{I}(2), \qquad M \cong \mathcal{P}(1), \qquad N \cong L \oplus M \cong \mathcal{P}(1) \oplus \mathcal{I}(2) \oplus \mathcal{P}(1).$$

As another example, consider Q is such that $\Delta_Q = \mathbb{A}_n$ with left-to-right arrow orientation, i.e.

$$Q = 1 \to \cdots \to n.$$

Ringel [Rin96, Appendix 2] presents the Auslander-Reiten quiver of Q, shown in Figure 3. Here, $M_{i,j+1} := \mathcal{S}(i) \oplus \mathcal{S}(i+1) \oplus \cdots \oplus \mathcal{S}(j)$. We also have the set of projective and injective representations given as;

$$\mathcal{P} = \{M_{1,n+1}, M_{2,n+1}, M_{3,n+1}, \ldots, M_{n-1,n+1}, M_{n,n+1}\},$$

$$\mathcal{I} = \{M_{1,n+1}, M_{1,n}, M_{1,n-1}, \ldots, M_{1,4}, M_{1,3}, M_{1,2}\}.$$

Naturally, we see that the bottom-most level of Γ_Q are the simple representations, the leading leftmost diagonal are the projective representations, and the rightmost diagonal (the last elements in each level) are the injective representations.

The last notion we cover here are the *meshes* which occur in the Auslander-Reiten quiver. In Figure 4 we see the four different meshes which can occur in an Auslander-Reiten quiver. These meshes represent the aforementioned almost split sequences and furthermore play an important role in the knitting algorithm.

4. The Knitting Algorithm

Here we present an algorithm to construct the Auslander-Reiten quiver for quivers such that Δ_Q is type $\mathbb{A}$ or $\mathbb{D}$. We encourage, as part of the expository nature of this paper, that readers interested in this material should explore QPA [22]—a package for GAP [22]. The following material (and much more) can be found within the QPA package, and in our case QPA can accelerate the computing of the Auslander-Reiten quiver by computing our representations. We will highlight this in Example 4.3.

To begin, we introduce the knitting algorithm itself:

The Knitting Algorithm. Let Q be a quiver, to construct the Auslander-Reiten quiver Γ_Q we follow the procedure;

(1) Compute the indecomposable projective representations

$$\mathcal{P}(1), \mathcal{P}(2), \ldots, \mathcal{P}(n).$$

(2) Draw an arrow $\mathcal{P}(i) \to \mathcal{P}(j)$ in Γ_Q if there exists an arrow $j \to i \in Q_1$. Position the $\mathcal{P}(i)$ to be on their own level.

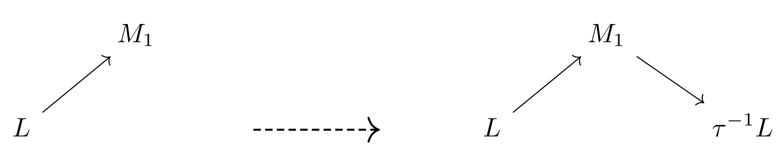
(3) (Knit) Complete the appropriate mesh such that

$$\underline{\dim} L + \underline{\dim} \tau^{-1} L = \sum_{i=1}^{2} \underline{\dim} M_i.$$

(4) Repeat (3) until there are negative integers in the dimension vector.

It is easy to see that the real difficulty of this algorithm lies within step (3), we focus now on explaining what it means to complete the mesh. First we recognize that the available meshes depend on Δ_Q. If $\Delta_Q = \mathbb{A}_n$ then there are three meshes (the first three shown in Figure 4), and completing them is illustrated below.

(4.1)



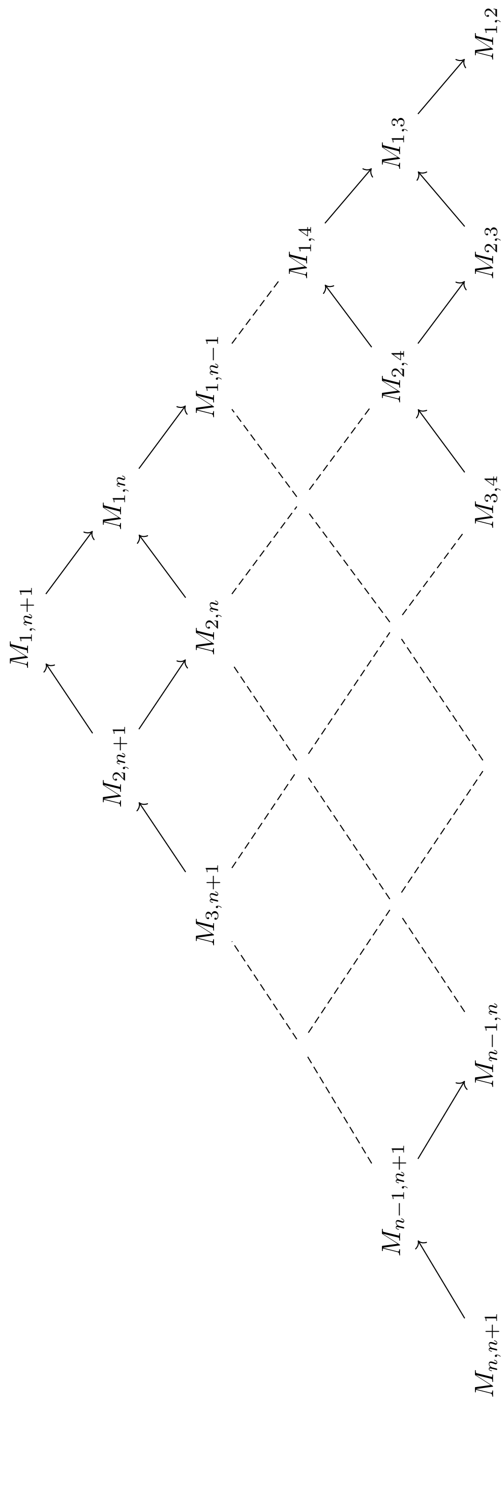


FIGURE 3. Auslander-Reiten quiver of $Q = 1 \to \cdots \to n$.

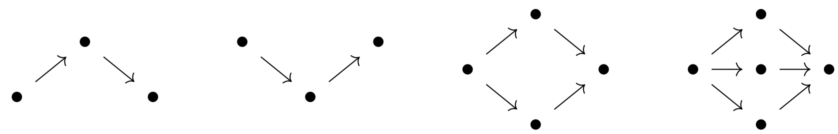
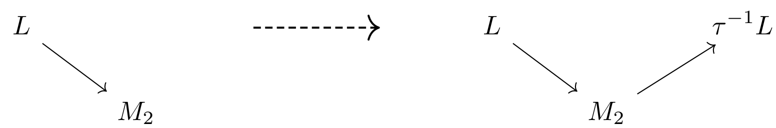
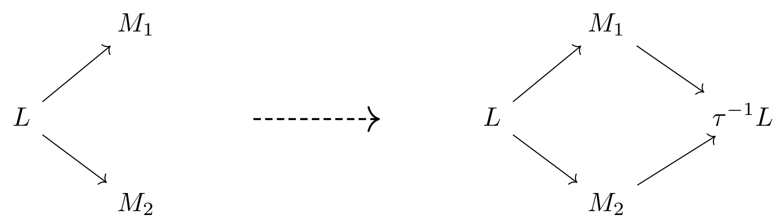


FIGURE 4. The four different types of meshes.

(4.2)

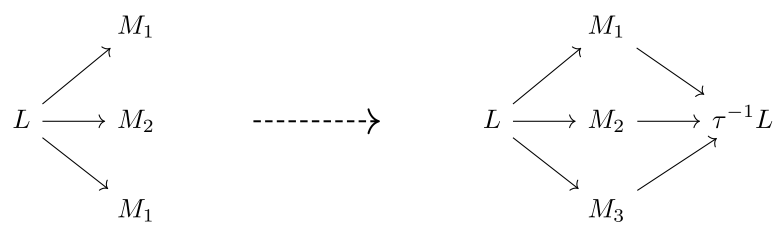


(4.3)



If $\Delta_Q = \mathbb{D}_n$ then there are four meshes, with the first three the same as type $\mathbb{A}_n$. The fourth mesh and its completion are shown below.

(4.4)



It follows that in the case of the fourth mesh, we must change the stopping criterion in step (3) of the algorithm. We modify it to be

$$\underline{\dim} L + \underline{\dim} \tau^{-1} L = \sum_{i=1}^{3} \underline{\dim} M_i.$$

4.1. **Coxeter Matrix Φ.** Regardless of which mesh we wish to complete we must compute a new representation $\tau^{-1}L$ (the symbol $\tau^{-1}L$ is given here since the traditional method of computing utilizes the *Auslander-Reiten translation*, which can be found in [AR75]). As noted at the end of subsection 2.1, it suffices to describe a representation in terms of its dimension vector. In this section we introduce and define the *Coxeter Matrix* Φ which will allow us to compute the dimension vector of $\tau^{-1}L$ and thus let us complete the mesh.

To do so, we need the *Cartan matrix* $C \in M_n(\mathbb{Z})$ defined as follows;

i. The ith column is the dimension vector of the indecomposable projective representation $\mathcal{P}(i)$.
ii. The ith row is the dimension vector of the indecomposable injective representation $\mathcal{I}(i)$.

Alternatively, for $C = (c_{ij})_{1\leq i,j\leq n}$, c_{ij} is the number of paths from j to i. Note that since in step (1) of the algorithm we compute all the indecomposable projective representations and thus constructing C becomes trivial.

Since Q is only of type $\mathbb{A}$ or $\mathbb{D}$, we have that it has no oriented cycles. Then we can renumber the vertices of Q such that, if there is a path from j to i, then $i \leq j$. Such an enumeration of vertices results in C being upper triangular (with diagonal entries given as just 1) and thus C is invertible.

We then define the Coxeter matrix Φ such that $\Phi = -C^T C^{-1}$, with natural inverse $\Phi^{-1} = -C(C^{-1})^T$. Then for an arbitrary representation M,

$$\Phi\underline{\dim}M = \underline{\dim}\tau M, \text{ if } M \text{ is not projective and } \Phi\underline{\dim}\mathcal{P}(i) = -\underline{\dim}\mathcal{I}(i).$$

And

$$\Phi^{-1}\underline{\dim}M = \underline{\dim}\tau^{-1}M, \text{ if } M \text{ is not injective and } \Phi^{-1}\underline{\dim}\mathcal{I}(i) = -\underline{\dim}\mathcal{P}(i).$$

Example 4.1. Suppose Q is as described in Example 2.5 with Γ_Q given in Example 3.1. Our three projective representations are given as

$$\underline{\dim}\mathcal{P}(1) = (1,1,0), \qquad \underline{\dim}\mathcal{P}(2) = (0,1,0), \qquad \underline{\dim}\mathcal{P}(3) = (0,1,1).$$

Then, the Cartan matrix is

$$C = \begin{bmatrix}\mathcal{P}(1) & \mathcal{P}(2) & \mathcal{P}(3)\end{bmatrix} = \begin{bmatrix}1 & 0 & 0\\ 1 & 1 & 1\\ 0 & 0 & 1\end{bmatrix}$$

with inverse

$$C^{-1} = \begin{bmatrix}1 & 0 & 0\\ -1 & 1 & -1\\ 0 & 0 & 1\end{bmatrix}.$$

Thus the Coxeter matrix is

$$\Phi = -C^T C^{-1} = \begin{bmatrix}0 & -1 & 1\\ 1 & -1 & 1\\ 1 & -1 & 0\end{bmatrix}$$

with inverse

$$\Phi^{-1} = -C(C^{-1})^T = \begin{bmatrix}-1 & 1 & 0\\ -1 & 1 & -1\\ 0 & 1 & -1\end{bmatrix}$$

Suppose that we want to compute $\underline{\dim}\tau^{-1}\mathcal{P}(2)$ (note that this is the same as computing $\underline{\dim}\tau^{-1}\mathcal{S}(2)$ since $\mathcal{P}(2) = \mathcal{S}(2)$), then we have $\Phi^{-1}(0,1,0) = (1,1,1)$ which is in fact $\underline{\dim}\mathcal{I}(2)$, where $\mathcal{I}(2)$ is the representation which completes the appropriate mesh.

4.2. **Examples of the Knitting Algorithm.** Here we provide two examples of using the Coxeter matrix in the knitting algorithm to compute the Auslander-Reiten quiver for two quivers; one of type $\mathbb{A}$ and the other of type $\mathbb{D}$. Both examples are taken from [Sch14, Chapter 3], but have been rewritten to illustrate the algorithm and use of the Coxeter matrix more clearly.

Example 4.2. Let Q be the quiver

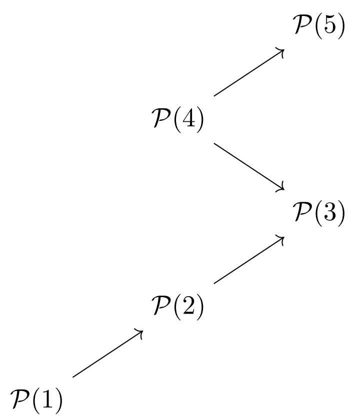
$$1 \longleftarrow 2 \longleftarrow 3 \longrightarrow 4 \longleftarrow 5\ .$$

Note that $\Delta_Q = \mathbb{A}_5$. We have the indecomposable projective representations;

$$\underline{\dim}\mathcal{P}(1) = (1,0,0,0,0), \qquad \underline{\dim}\mathcal{P}(2) = (1,1,0,0,0), \qquad \underline{\dim}\mathcal{P}(3) = (1,1,1,1,0),$$

$$\underline{\dim}\mathcal{P}(4) = (0,0,0,1,0), \qquad \underline{\dim}\mathcal{P}(5) = (0,0,0,1,1).$$

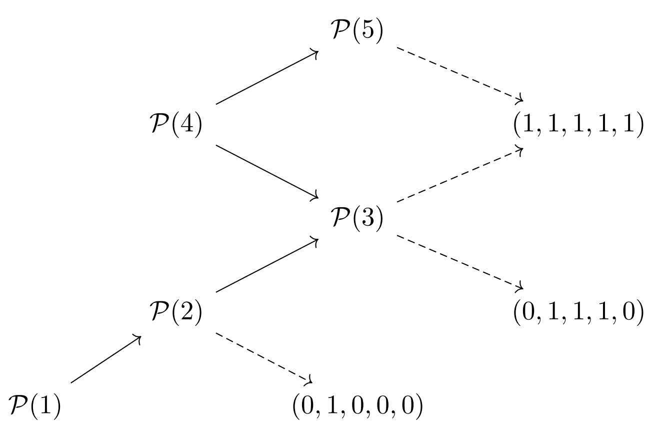
This completes step (1) of the algorithm, we now need to place the $\mathcal{P}(i)$ in Γ_Q along with their arrows. This gives us



In the same manner as above, we compute the Coxeter matrix (and its inverse) to be

$$\Phi = \begin{bmatrix} -1 & 1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & -1 & 0 \end{bmatrix} \qquad \Phi^{-1} = \begin{bmatrix} 0 & 0 & -1 & 1 & 0 \\ 1 & 0 & -1 & 1 & 0 \\ 0 & 1 & -1 & 1 & 0 \\ 0 & 1 & -1 & 1 & -1 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix}$$

We now start completing the mesh, beginning with the mesh consisting of $\mathcal{P}(1)$ and $\mathcal{P}(2)$ corresponding to (4.1). This gives $\underline{\dim}\tau^{-1}\mathcal{P}(1) = (0,1,0,0,0)$ with $(1,0,0,0,0)+(0,1,0,0,0) = (1,1,0,0,0) = \underline{\dim}\mathcal{P}(2)$ so the mesh is complete. Likewise, completing the appropriate meshes for $\mathcal{P}(2)$ and $\mathcal{P}(4)$ (meshes (4.1) and (4.3) respectively) gives us $\underline{\dim}\tau^{-1}\mathcal{P}(2) = (0,1,1,1,0)$ and $\underline{\dim}\tau^{-1}\mathcal{P}(4) = (1,1,1,1,1)$, one can easily see that the meshes are complete. These first three iterations give us an updated Auslander-Reiten quiver, we indicate the new meshes as dashed arrows.



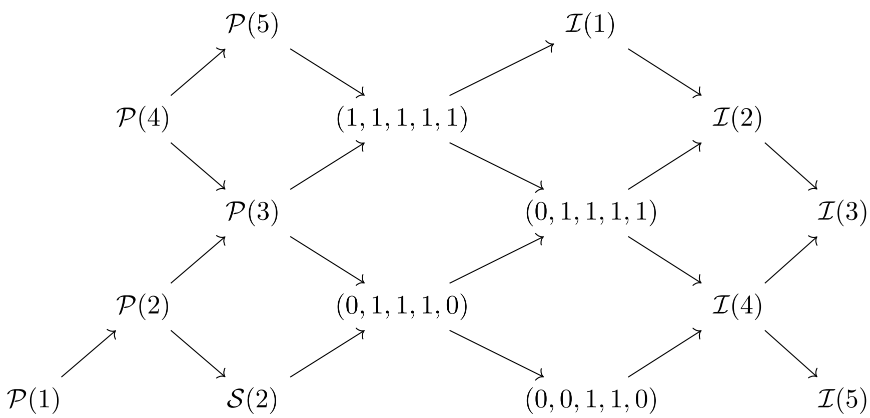
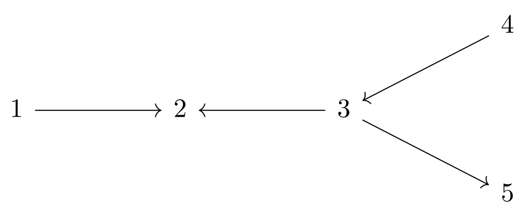


FIGURE 5. Auslander-Reiten quiver for Q as in Example 4.2.

Continuing on with this process we get the Auslander-Reiten quiver of Q to be as shown in Figure 5.

We know where to stop knitting when we reach injective representations, for instance if we tried to compute $\underline{\dim}\tau^{-1}\mathcal{I}(4)$ we would get $(0,0,0,-1,0)$ and there is a negative in the dimension vector, thus $\mathcal{I}(4)$ is the last element in that level.

Example 4.3. Let Q be the quiver

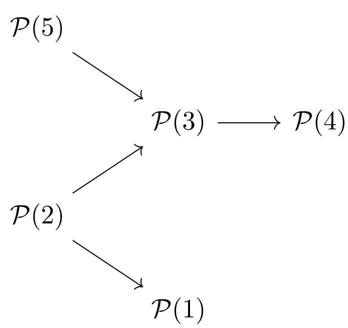


Note that $\Delta_Q = \mathbb{D}_5$. We have the indecomposable projective representations;

$$\underline{\dim}\mathcal{P}(1) = (1,1,0,0,0), \qquad \underline{\dim}\mathcal{P}(2) = (0,1,0,0,0), \qquad \underline{\dim}\mathcal{P}(3) = (0,1,1,0,1),$$

$$\underline{\dim}\mathcal{P}(4) = (0,1,1,1,1), \qquad \underline{\dim}\mathcal{P}(5) = (0,0,0,0,1).$$

We now place the $\mathcal{P}(i)$ in Γ_Q, noting that $\mathcal{P}(4)$ is placed on the same level as $\mathcal{P}(3)$.



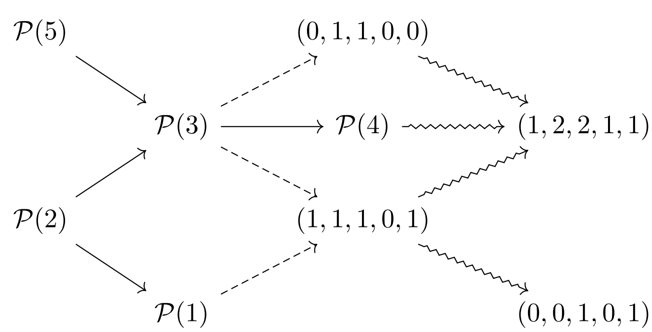
As mentioned earlier, much of this work can be accelerated by using the QPA package in GAP. To illustrate some use, we include some GAP code which mirrors the work we need to do.

```
gap> Q := Quiver(5, [ [1,2], [3,2], [3,5], [4,3] ]);;
gap> kQ := PathAlgebra(Rationals, Q);;
gap> P := IndecProjectiveModules(kQ);
[ <[ 1, 1, 0, 0, 0 ]>,
  <[ 0, 1, 0, 0, 0 ]>,
  <[ 0, 1, 1, 0, 1 ]>,
  <[ 0, 1, 1, 1, 1 ]>,
  <[ 0, 0, 0, 0, 1 ]> ]
gap> CoxeterInverse := Inverse(TransposedMat(CoxeterMatrix(kQ)));
[ [  -1,   1,   0,   0,   0 ],
  [  -1,   1,   0,  -1,   1 ],
  [   0,   1,   0,  -1,   1 ],
  [   0,   0,   1,  -1,   0 ],
  [   0,   1,   0,  -1,   0 ] ]
gap> # compute dim t^-1P(2)
gap> CoxeterInverse * [0,1,0,0,0];
[ 1, 1, 1, 0, 1 ]
gap> # compute dim t^-1P(5)
gap> CoxeterInverse * [0,0,0,0,5];
[ 0, 1, 1, 0, 0 ]
```

This reveals the first two new elements in Γ_Q ($\mathcal{P}(2)$ is mesh type (4.3) and $\mathcal{P}(5)$ is mesh type (4.2)). Then from here we can compute $\underline{\dim}\tau^{-1}\mathcal{P}(3)$ which is our first example of mesh (4.4).

```
gap> # compute dim t^-1P(3)
gap> CoxeterInverse * [0,1,1,0,1];
[ 1, 2, 2, 1, 1 ]
```

We illustrate the first iteration of the algorithm in dashed arrows and the second in squiggly arrows.



Finishing the algorithm then results in the Auslander-Reiten quiver of Q being as shown in Figure 6.

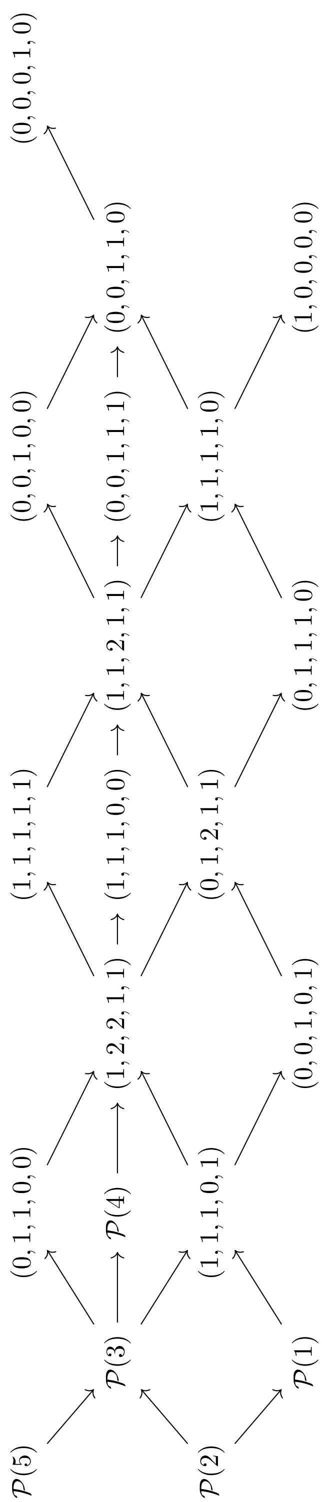


Figure 6. Auslander-Reiten quiver for Q as in Example 4.3.

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