A PROOF OF GABRIEL'S THEOREM

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Originally presented by Gabriel in [Gab72] and re-proven in [BGP73], Gabriel's Theorem is an incredibly powerful result connecting the study of quivers to that of finite algebras. In this small note we present the proof, which is taken more or less from [Sch14]. The theorem is as stated below.

Theorem 1 (Gabriel)**.** *Let Q be a connected quiver.*

(1) *Then Q is of finite representation type if and only if the underlying Dynkin diagram, Δ_Q, of Q is of Dynkin type A, D, or E.*
(2) *If Q has underlying Dynkin type A, D, or E then the dimension vector induces a bijection from isoclasses of indecomposable representations of Q to the set of positive roots:*

$$\Psi : \operatorname{ind} Q \to \Phi_+ \qquad \Psi : M \mapsto \underline{\dim} M.$$

Prior to completeing the proof, we need to develop the tools and mechanisms which we will utilize. Recall that a quiver $Q = (Q_0, Q_1, s, t)$ is a quadruple of a set of vertices and a set of arrows between vertices (Q_0 and Q_1 respectively), as well as a two set mapping functions associating the *source* and *target* of an arrow. We define a representation of a quiver by $M = (M_i, \varphi_\alpha)$, in which for each vertex we associate a k-vector space and each arrow becomes a linear map. We may define the dimension vector $\underline{\dim} M = (\dim M_i)$ to be the n-tuple of dimensions of each vector space of a representation, where n is the number of vertices in Q.

Let Q be a quiver without orientated cycles and n the number of vertices of Q. Fix some $\mathbf{d} = (d_i) \in \mathbb{Z}^n_{\geqslant 0}$, we call $\mathbf{d}$ the dimension vector and define the space $E_{\mathbf{d}} := \{M \mid \underline{\dim} M = \mathbf{d}\}$. It follows immediately that

$$E_{\mathbf{d}} = \bigoplus_{\alpha \in Q_0} \operatorname{Hom}_k(k^{d_{s(\alpha)}}, k^{d_{t(\alpha)}})$$

We now define the group

$$G_{\mathbf{d}} := \prod_{i \in Q_0} \mathrm{GL}_{d_i}(k).$$

Such a group acts on $E_{\mathbf{d}}$ via conjugation; more precisely, if $g = (g_i) \in G_{\mathbf{d}}$, $M = (M_i, \varphi_\alpha) \in E_{\mathbf{d}}$, and $i \xrightarrow{\alpha} j$ is an arrow in Q then we have that $(g \cdot \varphi)_\alpha = g_j \varphi_\alpha g_i^{-1}$.

We denote the orbit of a representation M under this action by $\mathcal{O}_M := \{g \cdot M \mid g \in G_{\mathbf{d}}\}$, while our motivation for the notion of orbits may seem unknown the following lemma illustrates the relevancy.

Lemma 2. *The orbit $\mathcal{O}_M$ is the isoclass of the representation M, that is,*

$$\mathcal{O}_M := \{M' \in \operatorname{rep} Q \mid M \cong M'\}.$$

Proof. Suppose that $M = (M_i, \varphi_\alpha)$ and $M' = (M'_i, \varphi'_\alpha)$ are in the same orbit, then there exists some $g \in G_{\mathbf{d}}$ such that $g \cdot M = M'$. That is, for each arrow $i \xrightarrow{\alpha} j$ in Q the following diagram commutes:

$$\begin{array}{ccc} M_i & \xrightarrow{\varphi_\alpha} & M_j \\ {\scriptstyle g_i}\downarrow & & \downarrow{\scriptstyle g_j} \\ M'_i & \xrightarrow[\varphi'_\alpha]{} & M'_j \end{array}$$

Therefore, g is a morphism of representations, moreover since each g_i is an element of $\mathrm{GL}_{d_i}(k)$ we have that it is invertible and thus an isomorphism. That is, $M \cong M'$. For the same argument, it follows immediately that if $M \cong M'$ then there is a $g \in G_{\mathbf{d}}$ such that $g \cdot M = g(M) = M'$. □

We know introduce the notions of a quadratic form a quiver Q, for Q a quiver without orientated cycles let $q(\mathbf{x})$ be the corresponding quadratic form of Q. We define it as follows

$$q(x_1, \dots, x_n) = \sum_{i \in Q_0}^{n} x_i^2 - \sum_{\alpha \in Q_1} x_{s(\alpha)} x_{t(\alpha)}$$

and associate with $q(x)$ the symmetric bilinear form

$$(\mathbf{x}, \mathbf{y}) = q(\mathbf{x} + \mathbf{y}) - q(\mathbf{x}) - q(\mathbf{y}).$$

Associated with q we have the notion of roots, both positive and negative, and furthermore real and imaginary. Let $x \in \mathbb{Z}^n \setminus \{0\}$, then x is a *real root* (resp. *imaginary root*) if $q(x) = 1$ (resp. $q(x) = 0$). It follows that every root x of q is of the form $a_1 e_1 + \cdots + a_n e_n$, i.e. $x = (a_1, \dots, a_n)$, if $a_i \geq 0$ (resp. $a_i \leq 0$) for $1 \leq i \leq n$ then x is a *positive root* (resp. *negative root*). We denote the set of all roots by Φ, the set of all positive roots Φ_+, and the set of all negative roots Φ_-.

The following proposition relates the dimension of the orbit to the relevant quadratic form.

Proposition 3. *Let Q be a connected quiver, $M \in \operatorname{rep} Q$, $\underline{\dim} M = \mathbf{d}$. Then,*

$$\operatorname{codim} \mathcal{O}_M = \dim \operatorname{End}(M) - q(\mathbf{d}) = \dim \operatorname{Ext}^1(M, M).$$

Proof. We have that $\dim \mathcal{O}_M = \dim G_{\mathbf{d}} - \dim \operatorname{Aut}(M)$, where $\operatorname{Aut}(M)$ is an open subgroup of $\operatorname{End}(M)$, so $\dim \operatorname{Aut}(M) = \dim \operatorname{End}(M)$. Moreover, $\dim G_{\mathbf{d}} = \sum_{i \in Q_0} d_i^2$ since $\dim \mathrm{GL}_{d_i} = d_i^2$. Thus

$$\operatorname{codim} \mathcal{O}_M = \dim E_{\mathbf{d}} - \dim \mathcal{O}_M = \underbrace{\sum_{\alpha \in Q_1} d_{s(\alpha)} d_{t(\alpha)} - \sum_{i \in Q_0} d_i^2}_{-q(\mathbf{d})} + \dim \operatorname{End}(M).$$

The second equality follows from a further result stating that $q(\mathbf{d}) = \dim \operatorname{Hom}(M, M) - \dim \operatorname{Ext}^1(M, M)$. □

Corollary. *If $q(\mathbf{d}) \leq 0$ then there are infinitely many isoclasses of representations of Q with dimension vector $\mathbf{d}$.*

Proof. Let $\mathbf{d}$ be as desired and $M \in \operatorname{rep} Q$ such that $\underline{\dim} M = \mathbf{d}$. Then $\operatorname{codim} \mathcal{O}_M \geq \dim \operatorname{End}(M) \geq 1$ implies the dimension of $E_{\mathbf{d}}$ is strictly greater than the dimension of any orbit $\mathcal{O}_M$. This then implies that the number of orbits is infinite, and thus the statement holds. □

Equipped with these results and tools, we can now begin the proof of Gabriel's Theorem. For our proof, we first prove (2) and then (1).

Proof of Theorem 1 (2). To prove our statement we show that Ψ is well-defined, then that Ψ is injective, and lastly that Ψ is surjective.

Let M be an indecomposable representation of Q, we need to show that $q(\underline{\dim} M) = 1$, of which it suffices to show $\operatorname{End} M \cong k$ and $\dim \operatorname{Ext}^1(M, M) = 0$. We first show that $\operatorname{End} M \cong k$, we proceed by induction on the dimension of M. If M is a simple representation, then it follows immediately. Suppose M has dimension strictly greater than 1, since M is indecomposable this implies that for all $f \in \operatorname{End} M$, $f = \lambda 1_M + g$ where $\lambda \in k$ and $g \in \operatorname{End} M$ is a nilpotent endomorphism. Since g is nilpotent, without loss of generality, we assume that $g^2 = 0$, moreover we choose g such that the dimension of the image of g is minimal. Then, $\operatorname{im} g \subset \ker g$ therefore there exists some indecomposable subrepresentation L such that $\operatorname{im} g \cap L$ is non-zero.

Let $\pi : \ker g \to L$ be the canonical projection and i the non-zero morphism given by the $\mathrm{incl} : \operatorname{im} g \to \ker g$ and π. That is,

$$\operatorname{im} g \xrightarrow{\ \mathrm{incl}\ } \ker g \xrightarrow{\ \pi\ } L, \qquad i = \pi \circ \mathrm{incl} : \operatorname{im} g \to L$$

This implies the composition $M \xrightarrow{g} \operatorname{im} g \xrightarrow{i} L \xrightarrow{\mathrm{incl}} M$ is a non-zero endomorphism whose square is zero. Then, the image is $i(\operatorname{im} g)$ and since g is taken to be minimal we have that $\dim i(img) \geq \dim \operatorname{im} g$ and thus i is injective. So the short exact sequence

$$0 \longrightarrow \operatorname{im} g \xrightarrow{\ i\ } L \longrightarrow \operatorname{coker} i \longrightarrow 0$$

can then have the $\operatorname{Hom}(-, L)$ functor applied to it and gives the following surjective morphism

$$\operatorname{Ext}^1(L, L) \longrightarrow \operatorname{Ext}^1(\operatorname{im} g, L) \longrightarrow 0$$

Then, by induction, $\dim \operatorname{Hom}(L, L) = 1$, and q is positive definite thus $\dim \operatorname{Ext}^1(L, L) = 0$ so the above equation shows that $\operatorname{Ext}^1(\operatorname{im} g, L) = 0$.

Consider the commutative diagram, whose rows are exact, and the bottom row is a push out of the top row along the morphism π.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \ker g & \xrightarrow{u} & M & \xrightarrow{g} & \operatorname{im} g & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle j_2} & & \| & & \\ 0 & \longrightarrow & L & \xrightarrow[j_1]{} & X & \longrightarrow & \operatorname{im} g & \longrightarrow & 0 \end{array}$$

Since $\operatorname{Ext}^1(\operatorname{im} g, L) = 0$ this implies that the bottom row splits so there exists some morphism $h : X \to L$ such that $hj_1 = 1_L$. Let $\nu : L \to \ker g$ be the inclusion of the direct summand, so $\pi\nu = 1_L$. We then construct $hj_2 : M \to L$ and $u\nu : L \to M$ such that $hj_2 u\nu = hj_1 \pi\nu = 1_L 1_L = 1_L$ and thus L is a direct summand of M. Thus, M is indecomposable so L must be either 0 or M. However, $L \neq 0$ since $\operatorname{im} g \cap L$ is non-zero and $L \neq M$ since $L \subset \ker g$ and $g \neq 0$. Therefore we arrive at a contradiction and $\dim \operatorname{End}(M) = 1$, q is positive definite, $\dim \operatorname{Ext}^1(M, M) = 0$, and $q(\underline{\dim} M) = 1$. Hence $\underline{\dim} M$ is a positive root and Ψ is well-defined.

We now show that Ψ is injective. Let $M, M' \in \operatorname{rep} Q$ such that they are both indecomposable and $\underline{\dim} M = \underline{\dim} M'$. We know that for Dynkin types A, D, and E the indecomposable representations have no self-extensions. Therefore, the orbits $\mathcal{O}_M$ and $\mathcal{O}_{M'}$ both have codimension zero, which occurs when $M \cong M'$. This shows that Ψ is injective.

Now we show that Ψ is surjective. Let Q be of Dynkin type A, D, or E, $\mathbf{d}$ a positive root, $M \in \operatorname{rep} Q$ such that $\underline{\dim} M = \mathbf{d}$ and $\mathcal{O}_M$ of maximum dimension in $E_{\mathbf{d}}$. We need to show that M is indecomposable. Let $M = M_1 \oplus M_2$, we will first show that $\operatorname{Ext}^1(M_1, M_2) = \operatorname{Ext}^1(M_2, M_1) = 0$. Suppose that $\operatorname{Ext}^1(M_1, M_2) \neq 0$, then this implies that there exists a non-split short exact sequence of the form

$$0 \longrightarrow M_2 \longrightarrow E \longrightarrow M_1 \longrightarrow 0$$

here $\underline{\dim} E = \underline{\dim} M$. Then a previous result implies that $\dim \mathcal{O}_M < \dim \mathcal{O}_E$, a contradiction of the maximality of $\mathcal{O}_M$. Thus, $\operatorname{Ext}^1(M_1, M_2) = 0$ and by symmetry we see that $\operatorname{Ext}^1(M_2, M_1) = 0$. Since $q(\mathbf{d}) = \dim \operatorname{Hom}(M, M) - \dim \operatorname{Ext}^1(M, M)$, we see that

$$1 = q(\mathbf{d}) = \dim \operatorname{Hom}(M_1 \oplus M_2, M_1 \oplus M_2) \geq 2$$

and arrive at a contradiction. Thus M is indecomposable, $\Psi(M) = \mathbf{d}$, and Ψ is surjective. □

We can now prove part (1).

Proof of Theorem 1 (1). Suppose Q is not of Dynkin type A, D, or E, then there exists some $\mathbf{d} \neq 0$ such that $q(\mathbf{d}) \leq 0$, so by the Corollary there are infinitely many isoclasses of representations with dimension vector $\mathbf{d}$. Each representation is a finite direct sum of indecomposable representations, therefore the number of isoclasses of indecomposable representations is infinite. This shows (1) and thus Gabriel's Theorem holds. □

References

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